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The orbifolds of permutation-type as physical string systems at multiples of $c = 26$ I. Extended actions and new twisted world-sheet gravities

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Abstract: This is the first in a series of papers in which I investigate the orbifolds of permutation-type as candidates for *new physical string systems* at multiples of critical closed-string central charges. Examples include the bosonic orientation orbifolds, the bosonic permutation orbifolds and others, as well as superstring extensions. In this paper I use the extended (twisted) Virasoro algebras of these orbifolds to construct the corresponding extended action formulations of the twisted open- and closed-string sectors of all the bosonic orbifolds at $\hat{c} = 26K$. The extended actions exhibit a large set of *new twisted world-sheet gravities*, whose extended diffeomorphism groups clearly indicate that the associated operator-string theories can be free of negative-norm states at higher central charge.

Keywords: Space-Time Symmetries, Conformal and W Symmetry, Conformal Field Models in String Theory, Classical Theories of Gravity.

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1. Introduction

At the level of examples, current-algebraic conformal field theory [1–11] and orbifold theory [11–22] are almost as old as string theory itself [23–26]. It is only in the last few years however that the orbifold program [27–37] has in large part completed the local description of the general closed-string orbifold conformal field theory $A(H)/H$. The program constructs all orbifolds at once, using the principle of local isomorphisms [29–32, 37] to map the H-symmetric CFT $A(H)$ into the twisted sectors of $A(H)/H$. For the reader interested in particular topics, the following list may be helpful:

- the twisted current algebras and stress tensors of the general current-algebraic orbifold [29–31],

- the twisted affine-primary fields, twisted operator algebras and twisted KZ equations[1] of all WZW orbifolds [32, 33, 35, 36],

- the world-sheet action formulation of all WZW and coset orbifolds in terms of so-called group orbifold elements with diagonal monodromy [32–36],

- the action formulation and twisted Einstein equations of a large class of sigma-model orbifolds [37],

- the general free-bosonic avatars of these constructions on abelian g and the explicit form of their twisted vertex operators [33, 35, 37].

A pedagogical review of the program is included in ref. [36]. Recent progress at the level of characters has been reported in refs. [27, 39]. Complimentary discussions of WZW twist fields and twisted free-bosonic vertex operators are found in refs. [40, 41] and [42] respectively.

In a second section, the technology of the orbifold program has also been applied to general twisted *open-string* conformal field theory:

- the WZW orientation orbifolds, including their branes and twisted open-string KZ equations [43],

- the action formulation, branes and twisted Einstein equations of WZW, coset and sigma-model orientation orbifolds [44],

- the open-string WZW orbifolds [45],

- the general twisted open WZW string, including all T-dualizations, branes, non-commutative geometry and twisted open-string KZ systems [46],

- the general twisted boundary-state equation of twisted open WZW strings [33, 45, 46],

- free-bosonic avatars of these constructions on abelian g [43, 45, 46].

The general orientation-orbifold CFT, which is constructed by twisting world-sheet orientation-reversing automorphisms, is apparently[2] a twisted generalization of orientifold CFT [47]. The construction of the general twisted open WZW string is a synthesis of closed-string orbifold theory and the theory of untwisted open WZW strings given in ref. [48]. Complementary discussions of the untwisted open WZW string are found in refs. [49, 50], and there is also a complementary literature on twisted branes (see e.g. ref. [51]) and permutation branes [52] on group manifolds.

The present series of papers opens a third, more phenomenological section in the orbifold program — in which some of the simplest conformal-field-theoretic results of the

[1]An abelian twisted KZ equation for the inversion orbifold $x \to -x$ was given earlier in ref. [38].

[2]I will have more to say about the relation between orientation orbifolds and orientifolds in succeeding papers of this series.

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program are applied to physical string theory: My goal here is to ask whether simple *orbifolds of permutation-type* can describe *new physical string systems* at higher central charge, for example the values

$$\hat{c} = 26K, \quad K = 2, 3, 4, \dots \tag{1.1}$$

which are found in the critical free-bosonic orbifolds of permutation-type. These orbifolds begin with one or more copies of the critical closed bosonic string, so that (choosing the closed-string critical dimension $d = 26$) many examples of this type have already been studied at the level of orbifold conformal field theory:

- the twisted open-string sectors of the free-bosonic orientation orbifolds [43, 44, 46] at $\hat{c} = 52$,
- the twisted closed-string sectors of the free-bosonic permutation orbifolds [27, 33, 35, 37] at any multiple of $c = 26$,
- the free-bosonic open-string permutation orbifolds and their T-duals [45, 46] at any multiple of $c = 26$
- the generalized [46] closed- and open-string permutation orbifolds at $\hat{c} = 26K$, including extra automorphisms which act uniformly on each copy of the original closed string (see also Subsection 4.2).

The corresponding critical superstring orbifolds of permutation-type can also be studied with central charges

$$\hat{c} = 10K, \quad K = 2, 3, \dots \tag{1.2}$$

or $(\hat{c}, \hat{\bar{c}}) = (26K, 10K)$ for heterotic type. Here the subject is not as well developed, but some discussion of permutation-orbifold superconformal field theory is found in refs. [22, 27].

The orbifolds of permutation-type have not previously been considered as candidates for physical string systems. One reason for this may be that the covariant formulation of the twisted sectors exhibit extra twisted time-like currents,[3] and hence extra sets of *negative norm states* (ghosts) associated with the higher central charge. It is important therefore to state the basic hypothesis which underlies this investigation: "Orbifoldization should not create negative-norm states" where there were none in the original symmetric theory. There are of course no ghosts in the untwisted sectors of these orbifolds — which are after all nothing but (symmetrizations of) decoupled copies of ordinary ghost-free strings [54] – and the orbifold program constructs the twisted sectors directly from the untwisted. On the basis of this hypothesis then, we are led to expect a natural, extended mechanism for ghost-decoupling in the twisted sectors of these orbifolds.

In this first installment of the series, I will find and study the classical precursor of this ghost-decoupling mechanism. The central observation is that each of the twisted sectors of these orbifolds contains an *extended (twisted) Virasoro algebra* [27, 55, 35, 43],

[3] Untwisted theories with two time-like dimensions have been considered in ref. [53].

which straightforwardly implies a *new extended (twisted) world-sheet gravity* in each sector. Because the new extended gravities are in 1-1 correspondence with the conjugacy classes of all the permutation groups, I will refer to them collectively as the *permutation gravities* (or P-gravities). The extended diffeomorphism groups of the permutation gravities clearly indicate that the corresponding twisted operator-string theories will exhibit new twisted BRST systems and new extended Ward identities — so that their string amplitudes can be free of negative-norm states. These and other topics at the operator level will be addressed in the succeeding papers of the series.

The world-sheet permutation gravities follow closely in the tradition of earlier extended world-sheet gravities, all of which are associated to extended Virasoro algebras and new critical central charges. I mention in particular a) the world-sheet supergravities [56] associated to superconformal algebras [57, 58], b) the W-gravities [59] associated to W-algebras [60], and c) the exotic world-sheet gravities of the generic affine-Virasoro constructions [61, 62] associated to K-conjugate (commuting) pairs of Virasoro algebras [2, 10, 11]. Similarly, one may expect new world-sheet *permutation supergravities* associated to the extended, twisted superconformal algebras [27] of superstring orbifolds of permutation-type.

The organization of this paper is as follows. I will consider two classes of examples, the twisted open-string sectors of the orientation orbifolds in section 2 and the twisted closed-string sectors of the permutation orbifolds in section 3. For both classes, I use the extended Virasoro algebras to construct the classical extended Hamiltonian systems — which then straightforwardly imply the *extended action formulations* of Polyakov-type. The final forms of these actions, including identification of the *extended (twisted) metrics*, are found in eqs. (2.52) and (3.28). The corresponding extended actions of Nambu-type are found in Subsections 2.8 and 3.4.

The open-string orientation-orbifold sectors are all governed by the simple case of $\mathbb{Z}_2$-permutation gravity, but the derivation in this case is complicated by the need to follow the boundary conditions (branes) of the twisted open strings. The replacement of boundary conditions by monodromies simplifies the derivation for the closed-string sectors of the permutation orbifolds, but here one encounters the systematics of the general world-sheet permutation gravity. In this case, I have also been able to find a simpler, complementary derivation of the extended actions (see Subsection 3.5) directly from the principle of local isomorphisms. For this reason, and because the permutation orbifolds are more familiar, the reader may wish to begin with section 3.

For generality, the results are worked out first for the orbifold CFT's at $\hat{c} = Kd$, where d is the number of free bosons in each copy of the untwisted closed-string CFT. The results for the critical orbifold-string theories at $d = 26$ and $\hat{c} = 26K$ are however easily obtained at any stage of the development and, using our quantitative knowledge of the permutation gravities, I finally return in section 4 to the conjectured properties of the critical orbifold-string theories at the operator level.

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2. $\mathbb{Z}_2$-permutation gravity in the orientation orbifolds

2.1 Extended virasoro algebra and extended hamiltonian

The general orientation orbifold [43, 44, 46] is constructed as

$$\frac{A(H_-)}{H_-}, \quad H_- = \mathbb{Z}_2(\text{world sheet}) \times H \tag{2.1}$$

where $A(H_-)$ is any closed-string CFT with central charge c and H_- is any symmetry group which includes world-sheet orientation-reversing automorphisms. Like orientifolds [47], each orientation orbifold contains an equal number of closed-and open-string sectors but, in contrast to orientifolds, the generic orientation-orbifold sector contains fractional modeing and the open-string sectors live at $\hat{c} = 2c$. The closed-string sectors (associated to the orientation-preserving subgroup of H_-) form an ordinary (space-time) orbifold [27–37] by themselves.

I concentrate here on the twisted open-string sectors (associated to the orientation-reversing automorphisms), each of which exhibits an order-two orbifold Virasoro algebra [27, 55, 35, 43]

$$\left[\hat{L}_u\left(m+\frac{u}{2}\right), \hat{L}_v\left(n+\frac{v}{2}\right)\right] = \left(m-n+\frac{u-v}{2}\right)\hat{L}_{u+v}\left(m+n+\frac{u+v}{2}\right) + \frac{\hat{c}}{12}\left(m+\frac{u}{2}\right)\left(\left(m+\frac{u}{2}\right)^2-1\right)\delta_{m+n+\frac{u+v}{2},0} \tag{2.2a}$$

$$\hat{c} = 2c, \quad \bar{u} = 0, 1. \tag{2.2b}$$

Here and below I assume the standard periodicity for all the spectral indices, so that $\bar{u}$ is the pullback of u to the fundamental range – and up and down indices u are equivalent. The (integral) Virasoro subalgebra of the extended algebra (2.2) is generated by $\{L_0(m)\}$, and section 2 of ref. [43] gives a simple explanation of the transition $c \to \hat{c} = 2c$ for the open-string sectors as well as the presence of the twisted generators $\left\{\hat{L}_1\left(m+\frac{1}{2}\right)\right\}$. In what follows, I will use essentially standard methods [63, 61] and the classical analogue of the extended Virasoro generators to construct the correspondingly-extended classical Hamiltonian of the general open-string orientation-orbifold sector.

I begin this discussion with the extended left- and right-mover stress tensors of the sector

$$\hat{\theta}_u^{\pm}(\xi,t) \equiv \frac{1}{2\pi}\sum_{m\in\mathbb{Z}} \hat{L}_u\left(m+\frac{u}{2}\right) e^{-i\left(m+\frac{u}{2}\right)(t\pm\xi)}, \quad \bar{u} = 0, 1 \tag{2.3a}$$

$$\hat{\theta}_u^{\pm}(\xi+2\pi,t) = (-1)^u \hat{\theta}_u^{\pm}(\xi,t) \tag{2.3b}$$

$$\hat{\theta}_u^{\mp}(\xi,t) = \hat{\theta}_u^{\pm}(-\xi,t) \tag{2.3c}$$

both of which are constructed from the same, single set of extended Virasoro generators. These are the extended stress tensors of the conformal field theory, whose form suffices to compute the following equal-time bracket algebra in the classical theory

$$\{\hat{\theta}_u^+(\xi,t),\hat{\theta}_v^+(\eta,t)\} = i(\partial_\xi-\partial_\eta)(\hat{\theta}_{u+v}^+(\eta,t)\delta_{\frac{u}{2}}(\xi-\eta)) \tag{2.4a}$$

$$= i\theta_{u+v}^+(\eta,t)\partial_\xi\delta_{\frac{u}{2}}(\xi-\eta) \tag{2.4b}$$

$$-i\hat{\theta}_{u+v}^+(\xi,t)\partial_\eta\delta_{\frac{v}{2}}(\eta-\xi)$$

$$\delta_{\frac{u}{2}}(\xi-\eta) = \frac{1}{2\pi}\sum_{m\in\mathbb{Z}} e^{-i\left(m+\frac{u}{2}\right)(\xi-\eta)} = e^{-i\frac{u}{2}(\xi-\eta)}\delta(\xi-\eta). \tag{2.4c}$$

from the classical analogue of the extended Virasoro algebra. The quantity in eq. (2.4c) is called a phase-modified delta function [37, 45, 46], and the other equal-time brackets of $\{\hat{\theta}_u^\pm\}$ follow from this result and eq. (2.3c).

In terms of the classical extended stress tensors, one may define the classical *extended Hamiltonian* $\hat{H}$ of the sector, as well as the generator $\hat{G}$ of time-independent gauge transformations

$$\hat{H} \equiv \int_0^{2\pi} d\xi \sum_u \hat{v}_+^u(\xi,t)\hat{\theta}_u^+(\xi,t), \quad \hat{G} \equiv \int_0^{2\pi} d\xi \sum_u \hat{\epsilon}_+^u(\xi)\hat{\theta}_u^+(\xi,t) \tag{2.5a}$$

$$\hat{v}_+^u(\xi+2\pi,t) = (-1)^u\hat{v}_+^u(\xi,t), \;\; \hat{\epsilon}_+^u(\xi+2\pi) = (-1)^u\hat{\epsilon}_+^u(\xi), \quad \bar{u}=0,1 \tag{2.5b}$$

where $\hat{v}$ and $\hat{\epsilon}$ are respectively the multipliers and the gauge parameters. Note that the densities here have trivial monodromy. We are interested however in the equivalent *open-string* forms of $\hat{H}$ and $\hat{G}$, which are integrated over the strip $0 \le \xi \le \pi$:

$$\hat{H} = \int_0^{\pi} d\xi \sum_u (\hat{v}_+^u(\xi,t)\hat{\theta}_u^+(\xi,t) + \hat{v}_-^u(\xi,t)\hat{\theta}_u^-(\xi,t)) \tag{2.6a}$$

$$\hat{G} = \int_0^{\pi} d\xi \sum_u (\hat{\epsilon}_+^u(\xi)\hat{\theta}_u^+(\xi,t) + \hat{\epsilon}_-^u(\xi)\hat{\theta}_u^-(\xi,t)) \tag{2.6b}$$

$$\hat{v}_-^u(\xi,t) = \hat{v}_+^u(-\xi,t), \quad \hat{\epsilon}_-^u(\xi) = \hat{\epsilon}_+^u(-\xi). \tag{2.6c}$$

In what follows, I will work directly with these open-string forms — including the following strip boundary conditions for all $n \in \mathbb{Z}_{\geq 0}$

$$\partial_\xi^n\hat{v}_-^u(0,t) = (-1)^n\partial_\xi^n\hat{v}_+^u(0,t), \quad \partial_\xi^n\hat{v}_-^u(\pi,t) = (-1)^{n+u}\partial_\xi^n\hat{v}_+^u(\pi,t) \tag{2.7}$$

which follow from eqs. (2.5b) and (2.6c). The same boundary conditions hold for the parameters $\{\hat{\epsilon}_\pm^u\}$, and indeed for the stress tensors $\{\hat{\theta}_u^\pm\}$ themselves.

I turn now to the Hamiltonian equations of motion, beginning with the extended classical constraints

$$\hat{\theta}_u^\pm(\xi,t) = 0 \;\;\leftrightarrow\;\; \{\hat{L}_u\left(m+\frac{u}{2}\right) = 0\}, \quad \bar{u}=0,1 \tag{2.8}$$

which are obtained by varying the multipliers in $\hat{H}$. These conditions generalize the standard classical Polyakov constraints $\hat{\theta}_0 = \hat{L}_0(m) = 0$, which are included here when $\bar{u}=0$. With these extended constraints included in the Hamiltonian formulation, we will not be surprised to recover them as well in the equivalent, extended action formulation (see Subsection 2.7) of each twisted sector.

The other Hamiltonian equations of motion are specified as usual by

$$\dot{\hat{A}} = i\{H, A\} \tag{2.9}$$

for all observables beyond the multipliers. With the brackets (2.4), this defines the time-dependence of the *gauge-variant* stress tensors

$$\dot{\hat{\theta}}_u^{\pm} = \pm \sum_v \left[\partial_\xi (\hat{\theta}^{\pm}_{u+v} \hat{v}^v_{\pm}) + \hat{\theta}^{\pm}_{u+v} \partial_\xi \hat{v}^v_{\pm} \right] \tag{2.10}$$

which must reduce to the CFT stress tensors (2.3) in a certain gauge to be discussed below. Then the time dependence of the multipliers

$$\dot{\hat{v}}^u_{\pm} = \mp \sum_v \hat{v}^{u-v}_{\pm} \overleftrightarrow{\partial}_\xi \hat{v}^v_{\pm}, \quad \hat{A} \overleftrightarrow{\partial}_\xi \hat{B} \equiv \hat{A} \partial_\xi \hat{B} - (\partial_\xi \hat{A}) \hat{B} \tag{2.11}$$

is obtained from eq. (2.10) and the requirement that $\dot{\hat{H}} = 0$.

Similarly, the time-independent gauge transformations are specified as

$$\delta \hat{A} = i\{\hat{G}, \hat{A}\}, \quad \delta \hat{v}^u_{\pm} \equiv \mp \sum_v \hat{v}^{u-v}_{\pm} \overleftrightarrow{\partial}_\xi \hat{\epsilon}^v_{\pm} \tag{2.12a}$$

$$\delta \hat{\theta}^{\pm}_u = \pm \sum_v \left[\partial_\xi (\hat{\theta}^{\pm}_{u+v} \hat{\epsilon}^v_{\pm}) + \hat{\theta}^{\pm}_{u+v} \partial_\xi \hat{\epsilon}^v_{\pm} \right]. \tag{2.12b}$$

It follows that the extended Hamiltonian (2.6a) is *gauge-invariant*

$$\delta \hat{H} = 0 \tag{2.13}$$

under the gauge group associated to the extended Virasoro algebra.

Among possible gauge conditions, I mention first the (partially-fixed) *Polyakov gauge*

$$\hat{v}^u_{\pm} = \hat{v}^0_{\pm} \delta_{u,0 \bmod 2} : \quad \hat{H} = \int_0^\pi d\xi (\hat{v}^0_+ \hat{\theta}^+_0 + \hat{v}^0_- \hat{\theta}^-_0) \tag{2.14}$$

in which the Polyakov form of the Hamiltonian is recovered. On the other hand, the *conformal gauge*

$$\hat{v}^u_{\pm} = \delta_{u,0 \bmod 2} : \quad \hat{H} = \int_0^\pi d\xi (\theta^+_0 + \theta^-_0) = \hat{L}_0(0) \tag{2.15}$$

is a completely-fixed gauge which reproduces the extended stress tensors (2.3) and Hamiltonian of the twisted open-string CFT. In particular, the expected time dependence of the extended stress tensors

$$\partial_\mp \hat{\theta}^{\pm}_u = 0, \quad \bar{u} = 0, 1, \;\; \partial_\pm \equiv \partial_t \pm \partial_\xi \tag{2.16}$$

is obtained from eq. (2.10) in the conformal gauge.

The corresponding extended action formulation of each open-string orientation orbifold sector can now be obtained by the standard Legendre transformation

$$\hat{S} = \int dt \int_0^\pi d\xi \sum_u (\dot{\hat{x}}^{n(r)\mu u} \hat{p}_{n(r)\mu u} - \hat{v}_+^u \hat{\theta}_u^+ - \hat{v}_-^u \hat{\theta}_u^-) \tag{2.17}$$

from the phase-space sigma-model form of the extended stress tensors, where $\hat{x}$ and $\hat{p}$ are the twisted coordinates and momenta of each sector. Following the discussion of refs. [63, 61], we know that each extended action will exhibit a further-extended gauge invariance with world-sheet space- and *time*-dependent gauge parameters:

$$\hat{\epsilon}_\pm^u(\xi) \to \hat{\epsilon}_\pm^u(\xi, t), \quad \bar{u} = 0, 1 \tag{2.18a}$$

$$\delta \hat{v}_\pm^u = \dot{\hat{\epsilon}}_\pm^u \mp \sum_v \hat{v}_\pm^{u-v} \overleftrightarrow{\partial}_\xi \hat{\epsilon}_\pm^v, \quad \delta \hat{x}^{n(r)\mu u} = \{\hat{G}, \hat{x}^{n(r)\mu u}\}. \tag{2.18b}$$

These transformations are in fact a form of the infinitesimal extended diffeomorphisms of world-sheet $\mathbb{Z}_2$-*permutation gravity*, whose covariant form will be more transparent in coordinate space. To obtain an explicit form of the extended action however, we must choose a specific class of models.

2.2 The free-bosonic orientation orbifolds

Ref. [46] gives the phase-space sigma model description of the open-string sectors of the general WZW orientation orbifold, and it would be interesting to work out the extended actions[4] in this case. I confine the discussion here however to the simple case of the free-bosonic orientation orbifolds

$$\frac{\mathrm{U}(1)^d}{\mathbb{Z}_2(w.s.) \times H} \tag{2.19}$$

which are also discussed in that reference as a special case on abelian g. In these cases, the twisted open-string sectors live at central charge $\hat{c} = 2d$.

In the notation of eq. (2.19), the non-trivial element of $\mathbb{Z}_2(w.s.)$ permutes the left- and right-mover currents $J, \bar{J}$ of the untwisted closed-string CFT $\mathrm{U}(1)^d$ while the extra automorphisms H act uniformly on the left-and right-movers. More precisely, each open-string sector of the orientation orbifold is obtained by twisting the action of a world-sheet orientation-reversing automorphism

$$[J_a(m), J_b(n)] = [\bar{J}_a(m), \bar{J}_b(m)] = mG_{ab}\delta_{m+n,0} \tag{2.20a}$$

$$J_a(m)' = \omega_a{}^b \bar{J}_b(m), \quad \bar{J}_a(m)' = \omega_a{}^b J_b(m) \tag{2.20b}$$

$$\omega_a{}^c \omega_b{}^d G_{cd} = G_{ab}, \ \omega \in H \tag{2.20c}$$

$$m, n \in \mathbb{Z}, \quad a, b = 1 \ldots d. \tag{2.20d}$$

where the quantity G_{ab} is the tangent-space metric of the untwisted CFT, with inverse G^{ab}. At any stage in the discussion below, one may substitute the explicit form

$$\frac{\mathrm{U}(1)^{26}}{\mathbb{Z}_2(w.s.) \times H} : \quad G_{ab} = G^{ab} = \begin{pmatrix} -1 & 0 \\ 0 & 1\!\!1 \end{pmatrix}, \ a, b = 0, 1, \ldots 25 \tag{2.21}$$

[4]The CFT (or conformal gauge) action of each open-string WZW orientation-orbifold sector is known [44] in terms of group orbifold elements on the solid half cylinder

to obtain the results for the orientation orbifolds of the critical Minkowski-space closed string. In this case of course, all the twisted open-string sectors live at operator central charge $\hat{c} = 52$.

For each of these twisted open-string sectors, ref. [46] gives the form of the classical extended stress tensors in terms of the twisted currents of each sector:

$$\hat{\theta}^{\pm}_{u}(\xi) = \frac{1}{8\pi}\mathcal{G}^{n(r)\mu;n(s)\nu}\sum_{v}\hat{J}^{\pm}_{n(r)\mu v}(\xi)\hat{J}^{\pm}_{n(s)\nu,u-v}(\xi), \quad \bar{u} = 0,1 \tag{2.22a}$$

$$\{\hat{J}^{+}_{n(r)\mu u}(\xi), \hat{J}^{+}_{n(s)\nu v}(\eta)\} = 4\pi i\delta_{n(r)+n(s),0 \bmod \rho(\sigma)}\delta_{u+v,0 \bmod 2} \times\mathcal{G}_{n(r)\mu;-n(r),\nu}\partial_{\xi}\delta_{\frac{n(r)}{\rho(\sigma)}+\frac{u}{2}}(\xi-\eta) \tag{2.22b}$$

$$\hat{J}^{\pm}_{n(r)\mu u}(\xi,t) = \sum_{m}\hat{J}_{n(r)\mu u}\left(m+\frac{n(r)}{\rho(\sigma)}+\frac{u}{2}\right)e^{-i\left(m+\frac{n(r)}{\rho(\sigma)}+\frac{u}{2}\right)(t\pm\xi)} \tag{2.22c}$$

$$\hat{J}^{-}_{n(r)\mu u}(0,t) = \hat{J}^{+}_{n(r)\mu u}(0,t), \quad \hat{J}^{-}_{n(r)\mu u}(\pi,t) = e^{2\pi i\left(\frac{n(r)}{\rho(\sigma)}+\frac{u}{2}\right)}\hat{J}^{+}_{n(r)\mu u}(\pi,t). \tag{2.22d}$$

Here the quantity

$$\begin{aligned}\mathcal{G}_{n(r)\mu;n(s)\nu} &= \chi_{n(r)\mu}\chi_{n(s)\nu}U_{n(r)\mu}{}^{a}U_{n(s)\mu}{}^{b}G_{ab} \\ &= \delta_{n(r)+n(s),0 \bmod \rho(\sigma)}\mathcal{G}_{n(r)\mu;-n(r),\nu}\end{aligned} \tag{2.23}$$

is the twisted tangent-space metric of the sector, which is a duality transformation [29, 31, 32, 37] of the untwisted metric. The quantities $\{\chi\}$ are normalization constants and $\mathcal{G}^{\cdot}$ in the extended stress tensors is the inverse of $\mathcal{G}_{\cdot\cdot}$. The unitary eigenmatrices U in the duality transformation (2.23) are determined by the so-called H-eigenvalue problem [29, 31, 32, 37] of each automorphism ω (see eq. (2.20))

$$\omega_{a}{}^{b}(U^{\dagger})_{b}{}^{n(r)\mu} = (U^{\dagger})_{a}{}^{n(r)\mu}e^{-2\pi i\frac{n(r)}{\rho(\sigma)}}, \quad \omega\in H \tag{2.24a}$$

$$\bar{n}(r)\in(0,1,\ldots\rho(\sigma)-1) \tag{2.24b}$$

where $\rho(\sigma), n(r), \mu$ are the order and the spectral and degeneracy indices respectively of ω. Following convention in the orbifold program, all quantities are periodic $n(r)\to n(r)\pm\rho(\sigma)$ in the spectral indices, $\bar{n}(r)$ is the pullback of $n(r)$ to the fundamental region, and I have suppressed explicit sums over repeated indices $\{\bar{n}(r),\mu\}$ in the stress tensors.

For the phase-space formulation, we also need the quasi-canonical realization of the twisted currents [46]

$$\hat{J}^{+}_{n(r)\mu u} = 2\pi\hat{p}_{n(r)\mu u} + \mathcal{G}_{n(r)\mu;n(s)\nu}\partial_{\xi}\hat{x}^{n(s)\nu,-u} \tag{2.25a}$$

$$\hat{J}^{-}_{n(r)\mu u} = (-1)^{u+1}(2\pi\hat{p}_{n(r)\mu u} - \mathcal{G}_{n(r)\mu;n(s)\nu}\partial_{\xi}\hat{x}^{n(s)\nu,-u}) \tag{2.25b}$$

where $\{\hat{x}^{n(r)\mu u}\}$ and $\{\hat{p}_{n(r)\mu u}\}$ are the extended, twisted coordinates and momenta. Because of the extra label $\bar{u} = 0,1$, each twisted open-string sector has exactly $2d$ extended coordinates[5] in agreement with the central charge $\hat{c} = 2d$ of each of these sectors. The

[5]As an example, the automorphism $\omega = -1$ gives $\rho(\sigma) = 2, U = 1, \bar{n} = 1$ and $\mu = a$, so that this twisted sector has d coordinates $\{\hat{x}^{1a0}\}$ with DN boundary conditions and d coordinates $\{\hat{x}^{1a1}\}$ with NN boundary conditions. This and many other examples are further discussed in refs. [43, 44, 46].

complete quasi-canonical algebra of $\hat{x}$ and $\hat{p}$ (including the twisted non-commutative geometry of $\hat{x}$ with itself) is given in eq. (4.33) of ref. [46], but we will need here only the brackets of the twisted coordinates with the currents [46]:

$$\{\hat{J}^{+}_{n(r)\mu u}(\xi), \hat{x}^{n(s)\nu v}(\eta)\} = -2\pi i \delta_{n(r)\mu u}{}^{n(s)\nu v}(\delta_{\bar{y}(r,u)}(\xi-\eta)+(-1)^{u+1}\delta_{\bar{y}(r,u)}(\xi+\eta)) \tag{2.26a}$$

$$\{\hat{J}^{-}_{n(r)\mu u}(\xi), \hat{x}^{n(s)\nu v}(\eta)\} = -2\pi i \delta_{n(r)\mu u}{}^{n(s)\nu v}((-1)^{u+1}\delta_{-\bar{y}(r,u)}(\xi-\eta)+\delta_{-\bar{y}(r,u)}(\xi+u)) \tag{2.26b}$$

$$\bar{y}(r,u) = \frac{\bar{n}(r)}{\rho(\sigma)} + \frac{\bar{u}}{2}. \tag{2.26c}$$

These brackets, the extended Hamiltonian (2.6a) and the Hamiltonian equations of motion (2.9) suffice to compute the derivatives of the twisted coordinates:

$$\dot{\hat{x}}^{n(r)\mu u} = \frac{1}{2}\mathcal{G}^{n(r)\mu;n(s)\nu}\sum_{v}(\hat{v}^{v}_{+}\hat{J}^{+}_{n(s)\nu,v-u} + (-1)^{u+1}\hat{v}^{v}_{-}\hat{J}^{-}_{n(s)\nu,v-u}) \tag{2.27a}$$

$$\partial_\xi \hat{x}^{n(r)\mu u} = \frac{1}{2}\mathcal{G}^{n(r)\mu;n(s)v}\sum_{v}(\hat{J}^{+}_{n(s)\nu,-u} + (-1)^{u}\hat{J}^{-}_{n(s)\nu,-u}). \tag{2.27b}$$

The spatial derivative in (2.27b) follows easily from the phase-space realization of the twisted currents.

Finally, I give the extended infinitesimal gauge transformation of the twisted coordinates

$$\delta\hat{x}^{n(r)\mu u} = \frac{1}{2}\mathcal{G}^{n(r)\mu;n(s)\nu}\sum_{v}(\hat{\epsilon}^{v}_{+}\hat{J}^{+}_{n(s)\nu,v-u} + (-1)^{u+1}\hat{\epsilon}^{v}_{-}\hat{J}^{-}_{n(s)\nu,v-u}) \tag{2.28}$$

which follows (in parallel to $\dot{\hat{x}}$) from eq. (2.12a).

2.3 Coordinate-space, branes and twisted currents

We may now begin the passage to coordinate space, where world-sheet $\mathbb{Z}_2$-permutation gravity can be seen in covariant form.

I begin with the boundary conditions or *branes* at the ends of the twisted open strings

$$\dot{\hat{x}}^{n(r)\mu 0}(0) = \partial_\xi \hat{x}^{n(r)\mu 1}(0) = 0 \tag{2.29a}$$

$$\cos\left(\frac{n(r)\pi}{\rho(\sigma)}\right)\dot{\hat{x}}^{n(r)\mu u}(\pi) - i\sin\left(\frac{n(r)\pi}{\rho(\sigma)}\right)\sum_{v}\hat{v}^{v}_{+}(\pi)\partial_\xi\hat{x}^{n(r)\mu,u-v}(\pi) = 0 \tag{2.29b}$$

which follow from eq. (2.27) and the boundary conditions (2.7), (2.22d) on the multipliers and the currents. Not surprisingly, the branes are quite different at the two ends of the twisted string — and I note in particular that the branes at π depend on the boundary values of the multipliers (see also eq. (2.37e)).

To move further into coordinate space, one needs to solve for the momenta in terms of the time derivatives. The required result

$$\begin{aligned}
\hat{p}_{n(r)\mu u} &= \frac{1}{\pi}\mathcal{G}_{n(r)\mu;n(s)\nu}\sum_v (M^{-1}\partial_t - N\partial_\xi)_{uv}\hat{x}^{n(s)\nu v} && \text{(2.30a)}\\
M^{uv} &= M^{(u+v)},\ M^{(w)} \equiv \hat{v}_+^w + (-1)^w \hat{v}_-^w && \text{(2.30b)}\\
(M^{-1})_{uv} &= (M^{-1})_{(u+v)},\ M^{-1}_{(w)} = \gamma^{-1}(\hat{v}_-^w + (-1)^w \hat{v}_+^w) && \text{(2.30c)}\\
N_{uv} &= N_{(u+v)} && \text{(2.30d)}\\
2N_{(0)} &\equiv \gamma^{-1}((\hat{v}_+^0)^2 + (\hat{v}_-^1)^2 - (\hat{v}_-^0)^2 - (\hat{v}_+^1)^2), && \text{(2.30e)}\\
2N_{(1)} &\equiv \hat{v}_+^1\hat{v}_-^0 + \hat{v}_-^1\hat{v}_+^0 &&\\
\gamma &\equiv (\hat{v}_-^0 + \hat{v}_+^0)^2 - (\hat{v}_-^1 - \hat{v}_+^1)^2 && \text{(2.30f)}
\end{aligned}$$

follows after some algebra from eqs. (2.25) and (2.27). The additional relations

$$\sum_w M^{(w)}N_{(w+u)} = \frac{1}{2}(\hat{v}_+^u + (-1)^{u+1}\hat{v}_-^u), \quad \bar{u} = 0, 1 \tag{2.31}$$

are then obtained from the explicit form of the matrices M and N.

As a first application of eq. (2.30), the coordinate-space form of the twisted currents

$$\begin{aligned}
\hat{J}^+_{n(r)\mu u} &= 2\mathcal{G}_{n(r)\mu;n(s)\nu}\sum_v \left(M^{-1}\partial_t - \left(N - \frac{1}{2}\right)\partial_\xi\right)_{uv}\hat{x}^{n(s)\nu v} && \text{(2.32a)}\\
\hat{J}^-_{n(r)\mu u} &= 2(-1)^{u+1}\mathcal{G}_{n(r)\mu;n(s)\nu}\sum_v \left(M^{-1}\partial_t - \left(N + \frac{1}{2}\right)\partial_\xi\right)_{uv}\hat{x}^{n(r)\nu v} && \text{(2.32b)}
\end{aligned}$$

is easily obtained from the phase-space realization (2.25).

2.4 Extended Polyakov action and $\mathbb{Z}_2$-twisted permutation gravity

To obtain the explicit coordinate-space form of the extended Polyakov action (2.17) of these sectors, it is simplest to first express all quantities in terms of the twisted currents

$$\begin{aligned}
\hat{p}_{n(r)\mu u} &= \frac{1}{4\pi}(\hat{J}^+_{n(r)\mu u} + (-1)^{u+1}\hat{J}^-_{n(r)\mu u}) && \text{(2.33a)}\\
\hat{S} &= \int dt \int_0^\pi d\xi\, \hat{\mathcal{L}}_0 && \text{(2.33b)}\\
\hat{\mathcal{L}}_0 &= \sum_u (\dot{\hat{x}}^{n(r)\mu u}\hat{p}_{n(r)\mu u} - \hat{v}_+^u\hat{\theta}_u^+ - \hat{v}_-^u\hat{\theta}_u^-) && \text{(2.33c)}\\
&= \frac{1}{8\pi}\mathcal{G}^{n(r)\mu;n(s)\nu}\sum_{u,v}(-1)^{v+1}(\hat{v}_+^u\hat{J}^-_{n(r)\mu v}\hat{J}^+_{n(s)\nu,u-v} + (+\leftrightarrow -)) && \text{(2.33d)}
\end{aligned}$$

and then use the coordinate-space form (2.32) of the currents.

It is then straightforward to put the extended Polyakov action in the following "covariant" form

$$\hat{S} = \frac{1}{4\pi}\int dt \int_0^\pi d\xi \sum_{u,v} \tilde{h}^{mn}_{uv}\partial_m \hat{x}^{n(r)\mu u}\mathcal{G}_{n(r)\mu;n(s)\nu}\partial_n \hat{x}^{n(s)\nu v} \tag{2.34a}$$

$$\partial_m = (\partial_0, \partial_1) = \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial \xi}\right), \qquad \tilde{h}^{mn}_{uv} = \hat{h}^{mn}_{(u+v)} \tag{2.34b}$$

$$\tilde{h}^{00}_{(w)} \equiv 2M^{-1}_{(w)}, \qquad \tilde{h}^{10}_{(w)} = \tilde{h}^{01}_{(w)} \equiv -2N_{(w)} \tag{2.34c}$$

$$\tilde{h}^{11}_{(w)} \equiv -\frac{1}{2}M^{(-w)} + \sum_u (\hat{v}^u_+ + (-1)^{u+1}\hat{v}^u_-)N_{(w+u)} \tag{2.34d}$$

where the matrices M and N are defined in (2.30) and the identity (2.31) was used to simplify these results. All dependence on the four phase-space multipliers $\{\hat{v}^u_\pm, \bar{u} = 0, 1\}$ is now collected in what I will call (and later motivate) the *extended inverse metric density* of $\mathbb{Z}_2$-permutation gravity

$$\tilde{h}^{mn}_{uv} = \tilde{h}^{nm}_{uv} = \begin{pmatrix} \tilde{h}^{mn}_{(0)} & \tilde{h}^{mn}_{(1)} \\ \tilde{h}^{mn}_{(1)} & \tilde{h}^{mn}_{(0)} \end{pmatrix}, \qquad m, n \in (0, 1) \tag{2.35}$$

which then has *four* independent degrees of freedom.

As a check on the algebra, I record our coordinate-space results in the completely-fixed conformal gauge (2.15)

$$\hat{v}^u_\pm = \delta_{u,0 \bmod 2}, \qquad M^{-1}_{(u)} = \frac{1}{2}\delta_{u,0 \bmod 2}, \qquad N_{(u)} = 0 \tag{2.36a}$$

$$\tilde{h}^{mn}_{(0)} = \eta^{mn} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \tilde{h}^{mn}_{(1)} = 0 \tag{2.36b}$$

$$\hat{S} = \frac{1}{4\pi}\int dt \int_0^\pi d\xi \eta^{mn}\mathcal{G}_{n(r)\mu;n(s)\nu}\sum_u \partial_m \hat{x}^{n(r)\mu u}\partial_n \hat{x}^{n(s)\nu,-u} \tag{2.36c}$$

$$\dot{\hat{x}}^{n(r)\mu 0}(0) = \partial_\xi \hat{x}^{n(r)\mu 1}(0) = 0 \tag{2.36d}$$

$$\dot{\hat{x}}^{n(r)\mu u}(\pi) = i\tan\left(\frac{n(r)\pi}{\rho(\sigma)}\right)\partial_\xi \hat{x}^{n(r)\mu u}(\pi) \tag{2.36e}$$

$$\hat{J}^+_{n(r)\mu u} = \mathcal{G}_{n(r)\mu;n(s)\nu}\partial_+ \hat{x}^{n(r)\mu,-u} \tag{2.36f}$$

$$\hat{J}^-_{n(r)\mu u} = (-1)^{u+1}\mathcal{G}_{n(r)\mu;n(s)\nu}\partial_- \hat{x}^{n(r)\mu,-u} \tag{2.36g}$$

$$\hat{\theta}^+_u = \frac{1}{8\pi}\mathcal{G}_{n(r)\mu;n(s)\nu}\sum_v \partial_+ \hat{x}^{n(r)\mu v}\partial_+ \hat{x}^{n(s)\mu,-u-v} \tag{2.36h}$$

$$\hat{\theta}^-_u = \frac{1}{8\pi}(-1)^{u+1}\mathcal{G}_{n(r)\mu;n(s)\nu}\sum_v \partial_- \hat{x}^{n(r)\mu v}\partial_- \hat{x}^{n(r)\mu,-u-v} \tag{2.36i}$$

$$\Box \hat{x}^{n(r)\mu u} = 0, \quad \partial_\mp \hat{J}^\pm_{n(r)\mu u} = \partial_\mp \hat{\theta}^\pm_u = 0 \tag{2.36j}$$

where η is the flat world-sheet metric. This is exactly the description given in refs. [43, 44, 46] for the twisted open-string CFT's of the orientation orbifolds. Mixed boundary conditions such as eq. (2.36e), even at vanishing twisted B field, are well-known [43–46] in the orbifold program. The twisted mode expansions of the extended coordinates $\{\hat{x}\}$, as well as their twisted non-commutative geometries, are also given for the CFT's in ref. [46].

As another check on the algebra, note that the extended action reduces in the Polyakov gauge (2.14) to the ordinary Polyakov action [64] of each twisted sector

$$\hat{v}^1_{\pm} = \tilde{h}^{mn}_{(1)} = 0 \tag{2.37a}$$

$$\tilde{h}^{mn}_{(0)} = \sqrt{-h_{(0)}}\, h^{mn}_{(0)} = \frac{1}{\hat{v}^0_+ + \hat{v}^0_-}\begin{pmatrix} 2 & \hat{v}^0_- - \hat{v}^0_+ \\ \hat{v}^0_- - \hat{v}^0_+ & -2\hat{v}^0_+\hat{v}^0_- \end{pmatrix} \tag{2.37b}$$

$$h^{(0)}_{mn} \equiv \mathrm{sign}(\hat{v}^0_- + \hat{v}^0_+)e^{-\phi}\begin{pmatrix} \hat{v}^0_-\hat{v}^0_+ & \frac{1}{2}(\hat{v}^0_- - \hat{v}^0_+) \\ \frac{1}{2}(\hat{v}^0_- - \hat{v}^0_+) & -1 \end{pmatrix} \tag{2.37c}$$

$$h^{(0)}_{mp}h^{pn}_{(0)} = \delta^n_m, \quad h_{(0)} \equiv \det(h^{(0)}_{mn}) \tag{2.37d}$$

$$\hat{S} = \frac{1}{4\pi}\int dt \int_0^{\pi} d\xi \sqrt{-h_{(0)}}\, h^{mn}_{(0)}\mathcal{G}_{n(r)\mu;n(s)\nu}\sum_u \partial_m\hat{x}^{n(r)\mu u}\partial_n\hat{x}^{n()\nu,-u} \tag{2.37e}$$

where $h^{(0)}_{mn}$ is the Polyakov metric and ϕ is the Weyl degree of freedom. It is then clear that the "extended inverse metric density" $\tilde{h}^{mn}_{uv}$ in eq. (2.35) is a "two-component" extension of the inverse metric density $\sqrt{-h_{(0)}}\, h^{mn}_{(0)}$ of ordinary world-sheet gravity.

2.5 Extended (Z_2-twisted) diffeomorphisms

I turn next to the coordinate-space form of the extended diffeomorphisms of $\mathbb{Z}_2$-permutation gravity.

After some algebra, one finds that the extended Hamiltonian gauge transformations (2.18b) and (2.28) can be put in the covariant form:

$$\delta\hat{x}^{n(r)\mu u} = \sum_v \hat{\beta}^{mv}\partial_m\hat{x}^{n(r)\mu,u-v}, \quad \bar{u} = 0,1 \tag{2.38a}$$

$$\delta\tilde{h}^{mn}_{(u)} = \sum_v\{\partial_p(\hat{\beta}^{pv}\tilde{h}^{mn}_{(u+v)}) - (\partial_p\hat{\beta}^{mv})\tilde{h}^{pn}_{(u+v)} - (\partial_p\hat{\beta}^{nv})\tilde{h}^{pm}_{(u+v)}\}. \tag{2.38b}$$

The explicit form of the four extended diffeomorphism parameters $\{\hat{\beta}^{mu}, \bar{u} = 0,1\}$ is

$$\hat{\beta}^{tu} \equiv \sum_v(\hat{\epsilon}^v_+ + (-1)^v\hat{\epsilon}^v_-)M^{-1}_{(v-u)}, \quad \bar{u} = 0,1 \tag{2.39a}$$

$$\hat{\beta}^{\xi u} \equiv \frac{1}{2}(\hat{\epsilon}^u_+ + (-1)^{u+1}\hat{\epsilon}^u_-) - \sum_v(\hat{\epsilon}^v_+ + (-1)^v\hat{\epsilon}^v_-)N_{(v-u)} \tag{2.39b}$$

where the $\hat{\epsilon}$'s are the four gauge parameters of the extended Hamiltonian formulation. I remind that the $\hat{\epsilon}$'s and hence the $\hat{\beta}$'s have arbitrary world-sheet space-time dependence, thus comprising a true, *twisted doubling* of the standard world-sheet gravitational gauge degrees of freedom.

Using eqs. (2.33b), (2.34a) and (2.38), one then obtains the corresponding transformation of the extended action

$$\delta\hat{\mathcal{L}}_0 = \partial_m\left(\sum_w \hat{\beta}^{mw}\hat{\mathcal{L}}_w\right) \tag{2.40a}$$

$$\hat{\mathcal{L}}_w \equiv \frac{1}{4\pi}\sum_{u,v}\tilde{h}^{mn}_{(w+u+v)}\partial_m\hat{x}^{n(r)\mu u}\mathcal{G}_{n(r)\mu;n(s)\nu}\partial_n\hat{x}^{n(s)\nu v} \tag{2.40b}$$

$$\delta\hat{S} = \int dt\sum_w(\hat{\beta}^{\xi w}(\pi)\hat{\mathcal{L}}_w(\pi) - \hat{\beta}^{\xi w}(0)\hat{\mathcal{L}}_w(0)) \tag{2.40c}$$

where $\hat{\mathcal{L}}_w$ is the natural two-component extension of the action density $\hat{\mathcal{L}}_0$. The result in eq. (2.40c) requires that we study the coordinate-space boundary conditions in further detail.

The following boundary conditions on the $\mathbb{Z}_2$-gravitational structures

$$M^{-1}_{(1)}(0) = N_{(0)}(0) = N_{(u)}(\pi) = 0 \tag{2.41a}$$

$$\tilde{h}^{00}_{(1)}(0) = \tilde{h}^{11}_{(1)}(0) = \tilde{h}^{01}_{(0)}(0) = 0 \tag{2.41b}$$

$$\hat{\beta}^{\xi 0}(0) = \hat{\beta}^{\xi u}(\pi) = \beta^{t1}(0) = 0 \tag{2.41c}$$

$$\tilde{h}^{11}_{(u)}(\pi) = -\hat{v}^u_+(\pi), \quad \tilde{h}^{01}_{(u)}(\pi) = 0 \tag{2.41d}$$

follow from the original Hamiltonian boundary conditions (2.7) on $\hat{v}$ and $\hat{\epsilon}$, using the definitions of $M, N, \tilde{h}$ and β in eqs. (2.30), (2.34) and (2.39). These are not quite enough for the invariance of $\hat{S}$, but these conditions and the boundary conditions (2.29a) on the twisted coordinates at $\xi = 0$ suffice to show that

$$\hat{\mathcal{L}}_1(0) = 0 \;\rightarrow\; \delta\hat{S} = 0. \tag{2.42}$$

As expected from the Hamiltonian formulation, the extended action $\hat{S}$ is invariant under the extended ($\mathbb{Z}_2$-twisted) diffeomorphisms of $\mathbb{Z}_2$-permutation gravity.

2.6 The twisted coordinate equation of motion

It is instructive to vary the extended action (2.34a) by arbitrary infinitesimal variations $\delta\hat{x}$ of the extended coordinates. Then $\delta\hat{S} = 0$ gives the coordinate equations of motion:

$$\partial_m\left(\sum_w \tilde{h}^{mn}_{(u+w)}\partial_n\hat{x}^{n(r)\mu w}\right) = 0 \qquad \text{(bulk)} \tag{2.43a}$$

$$\sum_{u,v}\tilde{h}^{m1}_{(u+v)}\partial_m\hat{x}^{n(r)\mu u}\mathcal{G}_{n(r)\mu;n(s)\nu}\delta\hat{x}^{n(s)\nu v} = 0 \qquad \text{at } \xi = 0,\pi. \tag{2.43b}$$

The variational boundary conditions (2.43b) at the branes are in fact solved in the form

$$\sum_{u,v}\tilde{h}^{m1}_{(u+v)}\partial_m\hat{x}^{n(r)\mu u}\mathcal{G}_{n(r)\mu;n(s)\nu}\dot{\hat{x}}^{n(s)\nu v} = 0 \qquad \text{at } \xi = 0,\pi \tag{2.44}$$

by the following coordinate-space boundary conditions:

$$\xi = 0: \quad \dot{\hat{x}}^{n(r)\mu 0} = \partial_\xi \hat{x}^{n(r)\mu 1} = 0 \tag{2.45a}$$

$$\tilde{h}^{01}_{(0)} = \tilde{h}^{11}_{(0)} = 0 \tag{2.45b}$$

$$\xi = \pi: \quad \dot{\hat{x}}^{n(r)\mu u} + i \tan\left(\frac{n(r)\pi}{\rho(\sigma)}\right) \sum_w \tilde{h}^{11}_{(w)} \partial_\xi \hat{x}^{n(r)\mu, u-w} = 0 \tag{2.45c}$$

$$\tilde{h}^{01}_{(u)} = 0, \quad \bar{u} = 0, 1. \tag{2.45d}$$

These conditions are equivalent to those obtained earlier from phase space. For example the relation (2.45c) is nothing but the boundary condition (2.29b) written now with eq. (2.41d) in terms of the extended inverse metric density. The condition (2.44) at π is then solved by (2.45c) because the identity

$$\sum_{n(r),n(s)} \sum_{u,v,w} \tilde{h}^{11}_{(u+v+w)} \tilde{h}^{11}_{(w)} \tan\left(\frac{n(r)\pi}{\rho(\sigma)}\right) \partial_\xi \hat{x}^{n(r)\mu u} \mathcal{G}_{n(r)\mu;n(s)\nu} \partial_\xi \hat{x}^{n(s)\nu v} = 0 \tag{2.46}$$

holds by $n(r) \to -n(r)$ symmetry. For clarity I have here temporarily reinstated the implied sums over the spectral indices.

2.7 Identification of the extended metric

Following our discussion of the Polyakov gauge in Subsection 2.4, it is clear that the extended inverse metric density $\tilde{h}^{mn}_{(u)}$ must be further decomposed to obtain the true *extended metric* $\hat{h}^{(u)}_{mn}$ of $\mathbb{Z}_2$-permutation gravity. The correct decomposition is

$$\tilde{h}^{mn}_{(u)} = \sum_w \hat{h}^{mn}_{(u+w)} \hat{H}^{(w)}, \qquad \bar{u} = 0, 1, \quad m, n \in (0, 1) \tag{2.47a}$$

$$\sum_w \hat{h}^{mp}_{(u+w)} \hat{h}^{(w+v)}_{pn} = \delta^m_n \delta_{u-v, 0 \bmod 2} \tag{2.47b}$$

$$\hat{H}^{(u)} \equiv \frac{1}{2}\left(\sqrt{-\det(\textstyle\sum_w \hat{h}^{(w)}_{mn})} + (-1)^u \sqrt{-\det(\textstyle\sum_w (-1)^w \hat{h}^{(w)}_{mn})}\right) \tag{2.47c}$$

where $\hat{h}^{(u)}_{mn}$ and $\hat{h}^{mn}_{(u)}$ are respectively the extended metric and its inverse. The determinant in eq. (2.47c) is $\det(A_{mn}) = A_{00}A_{11} - (A_{01})^2$. Then it is straightforward to check that the extended, $\mathbb{Z}_2$-twisted diffeomorphisms

$$\delta \hat{h}^{(u)}_{mn} = \sum_w (\hat{\beta}^{pw} \partial_p \hat{h}^{(u-w)}_{mn} + \partial_m \hat{\beta}^{pw} \hat{h}^{(u-w)}_{pn} + \partial_n \hat{\beta}^{pw} \hat{h}^{(u-w)}_{pn}) \tag{2.48a}$$

$$\delta \hat{h}^{mn}_{(u)} = \sum_w (\hat{\beta}^{pw} \partial_p \hat{h}^{mn}_{(u+w)} - \partial_p \hat{\beta}^{mw} \hat{h}^{pn}_{(u+w)} - \partial_p \hat{\beta}^{nw} \hat{h}^{pm}_{(u+w)}) \tag{2.48b}$$

$$\delta \hat{H}^{(u)} = \partial_m \left(\sum_w \hat{\beta}^{mw} \hat{H}^{(u-w)}\right), \quad \bar{u} = 0, 1 \tag{2.48c}$$

are consistent and reproduce the extended diffeomorphisms (2.38b) of the extended inverse metric density. Correspondingly, any object which transforms like $\hat{H}^{(u)}$ in (2.48c) will be called an extended scalar density.

Note that the extended metric

$$\hat{h}_{mn}^{(u)} = \begin{pmatrix} \hat{h}_{00}^{(u)} & \hat{h}_{01}^{(u)} \\ \hat{h}_{10}^{(u)} & \hat{h}_{11}^{(u)} \end{pmatrix}, \quad \hat{u} = 0, 1 \tag{2.49}$$

has *six* independent degrees of freedom, while the extended inverse metric density has only four. This tells us that the extended metric contains *two* Weyl degrees of freedom. Indeed, I will argue in Subsection 2.9 that an appropriate parametrization of the extended metric in the (completely gauge-fixed) conformal gauge is

$$\hat{h}_{mn}^{(u)} = \frac{1}{2}\eta_{mn}(e^{-\hat{\phi}_0} + (-1)^u e^{-\hat{\phi}_1}) \tag{2.50}$$

where $\{\hat{\phi}_I,\ I = 0,1\}$ are the extended Weyl degrees of freedom and η_{mn} is the flat world-sheet metric. As a check on this form, one easily computes

$$\hat{H}^{(u)} = \frac{1}{2}(e^{-\hat{\phi}_0} + (-1)^u e^{-\hat{\phi}_1}) \tag{2.51a}$$

$$\hat{h}^{mn}_{(u)} = \frac{1}{2}\eta^{mn}(e^{\hat{\phi}_0} + (-1)^u e^{\hat{\phi}_1}) \tag{2.51b}$$

$$\to \tilde{h}^{mn}_{(u)} = \eta^{mn}\delta_{u,0 \text{ mod } 2} \tag{2.51c}$$

as required in the conformal gauge.

Identification of the extended metric also allows us to consider the $\mathbb{Z}_2$-gravitational equations of motion, which are obtained by arbitrary variation $\delta\hat{h}_{mn}^{(u)}$ of the final form of the extended action:

$$\hat{S} = \frac{1}{4\pi}\int dt \int_0^\pi d\xi \sum_{u,v,w} \hat{h}^{mn}_{(u+v+w)}\hat{H}^{(u)}\partial_m\hat{x}^{n(r)\mu v}\mathcal{G}_{n(r)\mu;n(s)\nu}\partial_n\hat{x}^{n(s)\nu w}. \tag{2.52}$$

Useful identities in this computation include the following

$$\sum_w \hat{H}^{-1}_{(u+w)}\hat{H}^{(w+v)} = \delta_{u-v,0 \text{ mod } 2} \tag{2.53a}$$

$$\delta\hat{H}^{(u)} = \frac{1}{2}\sum_{v,w}\hat{H}^{(u+v-w)}\hat{h}^{pq}_{(v)}\delta\hat{h}^{(w)}_{pq} \tag{2.53b}$$

$$\delta\hat{h}^{mn}_{(u)} = -\sum_{w,v}\hat{h}^{mp}_{(w)}\hat{h}^{qn}_{(v)}\delta\hat{h}^{(w+v-u)}_{pq} \tag{2.53c}$$

where the quantity $\hat{H}^{-1}$ is defined by eq. (2.53a). One then finds the extended $\mathbb{Z}_2$-gravitational stress tensor and equations of motion:

$$\hat{\theta}^{(u)}_{mn} \equiv \sum_{xyv}\hat{H}^{-1}_{(v)}\hat{h}^{(x)}_{mp}\hat{h}^{(y)}_{nq}\left(\delta\hat{S}/\delta\hat{h}^{(x+y-v-u)}_{pq}\right) \tag{2.54a}$$

$$= \frac{1}{2}\left(\hat{\mathcal{L}}^{(u)}_{mn} - \frac{1}{2}\sum_{v,w}\hat{h}^{(v)}_{mn}\hat{h}^{pq}_{(w)}\hat{\mathcal{L}}^{(w-v+u)}_{pq}\right) \tag{2.54b}$$

$$\hat{\mathcal{L}}^{(u)}_{mn} \equiv \frac{1}{4\pi}\sum_v \partial_m\hat{x}^{n(r)\mu v}\mathcal{G}_{n(r)\mu;n(s)\nu}\partial_n\hat{x}^{n(s)\nu,u-v} \tag{2.54c}$$

$$\sum_v \hat{h}^{mn}_{(u+v)}\hat{\theta}^{(v)}_{mn} = 0 \tag{2.54d}$$

$$\hat{\theta}^{(u)}_{mn} = 0, \quad \bar{u} = 0, 1. \tag{2.54e}$$

The extended trace conditions in eq. (2.54d) hold independent of the gravitational equations of motion in (2.54e).

In the conformal gauge, these results reduce to the following Weyl-field-independent forms

$$\eta^{mn}\hat{\theta}^{(u)}_{mn} = 0 \tag{2.55a}$$

$$\hat{\theta}^{(u)}_{mn} = \frac{1}{2}\left(\hat{\mathcal{L}}^{(u)}_{mn} - \frac{1}{2}\eta_{mn}\eta^{pq}\hat{\mathcal{L}}^{(u)}_{pq}\right) = 0, \quad \bar{u} = 0,1 \tag{2.55b}$$

after using the form (2.50) of the extended metric in the conformal gauge. As expected from the extended Hamiltonian description, these conditions are nothing but linear combinations of the classical *extended Virasoro constraints* of each twisted open-string CFT

$$\hat{\theta}^{\pm}_{u}(\xi,t) = 0, \quad \bar{u} = 0,1. \tag{2.56}$$

where the coordinate-space form of $\{\hat{\theta}^{\pm}_{u}\}$ was given in eq. (2.36).

2.8 Extended nambu action

The quantity $\hat{\mathcal{L}}^{(u)}_{mn}$ of the previous subsection is constructed entirely from the extended coordinates and transforms under the $\mathbb{Z}_2$-twisted diffeomorphisms like the extended metric:

$$\hat{\mathcal{L}}^{(u)}_{mn} \approx \sum_{v} \partial_m \hat{x}^{n(r)\mu v}\mathcal{G}_{n(r)\mu;u(s)\nu}\partial_n \hat{x}^{n(s)\nu,u-v} \tag{2.57a}$$

$$\delta\hat{x}^{n(r)\mu u} = \sum_{v}\hat{\beta}^{mv}\partial_m \hat{x}^{n(r)\mu,u-v} \tag{2.57b}$$

$$\delta\hat{\mathcal{L}}^{(u)}_{mn} = \sum_{v}(\hat{\beta}^{pv}\partial_p\hat{\mathcal{L}}^{(u-v)}_{mn} + \partial_m\hat{\beta}^{pv}\hat{\mathcal{L}}^{(u-v)}_{pn} + \partial_n\hat{\beta}^{pv}\hat{\mathcal{L}}^{(u-v)}_{pm}). \tag{2.57c}$$

This identifies $\mathcal{L}^{(u)}_{mn}$ as the natural candidate for the extended form of the induced world-sheet metric.

We are then led to consider the following extended action of Nambu-type

$$\hat{\hat{S}} \approx \int dt \int_0^{\pi} d\xi\, \hat{\hat{H}}^{(0)} \tag{2.58a}$$

$$\hat{\hat{H}}^{(u)} \equiv \frac{1}{2}\left(\sqrt{-\det(\textstyle\sum_v \hat{\mathcal{L}}^{(v)}_{mn})} + (-1)^u\sqrt{-\det(\textstyle\sum_v (-1)^v\hat{\mathcal{L}}^{(v)}_{mn})}\right) \tag{2.58b}$$

$$\delta\hat{\hat{H}}^{(0)} = \partial_p\left(\sum_u \hat{\beta}^{pu}\hat{\hat{H}}^{(-u)}\right) \tag{2.58c}$$

for each twisted open-string sector of $\mathrm{U}(1)^d/(\mathbb{Z}_2(w.s.)\times H)$. Eq. (2.58c) tells us in particular that the quantity $\hat{\hat{H}}^{(u)}$ is an extended scalar density and, using the boundary conditions (2.41) on the extended diffeomorphism parameters $\hat{\beta}$, one finds that this action is invariant under the extended diffeomorphisms when $\hat{\hat{H}}^{(1)}(0,t) = 0$.

For brevity, I will not pursue further analysis of these Nambu-like systems here. For future work, I note however that the four degrees of freedom of the extended diffeomorphisms should allow a "light-cone gauge" in which the number of independent (transverse)

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coordinate degrees of freedom is $2d-4$. See also my more detailed remarks on the extended Nambu actions of the permutation orbifolds in Subsection 3.4 and my concluding remarks in Subsection 4.1 on the corresponding operator theories at critical dimension $d=26$.

2.9 Finite invariance transformations

I collect here some preliminary remarks on the finite symmetries of the extended action (2.52) of Polyakov-type, including the extended Weyl invariance and the extended diffeomorphism group of $\mathbb{Z}_2$-permutation gravity.

The central step in this discussion is to define what corresponds to the "fields with twisted boundary conditions" [29, 37] of each sector:

$$\hat{h}^I_{mn} \equiv \sum_u (-1)^{Iu} \hat{h}^{(u)}_{mn}, \quad I=0,1 \tag{2.59a}$$

$$\hat{h}_I^{mn} \equiv \sum_u (-1)^{Iu} \hat{h}^{mn}_{(u)}, \quad \hat{h}_I^{mp}\hat{h}^I_{pn} = \delta_n^m \tag{2.59b}$$

$$\hat{H}^I \equiv \sum_u (-1)^{Iu} \hat{H}^{(u)} = \sqrt{-\det(\hat{h}^I_{mn})} \tag{2.59c}$$

$$\hat{x}^{n(r)\mu I} \equiv \sum_u (-1)^{Iu} \hat{x}^{n(r)\mu u}, \quad \hat{x}^{n(r)\mu u} = \frac{1}{2}\sum_I (-1)^{uI} \hat{x}^{n(r)\mu I}. \tag{2.59d}$$

All these transformations are invertible, as shown explicitly for the twisted coordinates in eq. (2.59d).

After some algebra, one finds that the new fields *diagonalize* the extended action and its diffeomorphisms

$$\hat{S} = \sum_I \int dt_I \int_0^\pi d\xi_I \, \hat{H}^I \hat{h}_I^{mn} \partial_m \hat{x}^{n(r)\mu I} \mathcal{G}_{n(r)\mu;n(s)\nu} \partial_n \hat{x}^{n(s)\nu I} \tag{2.60a}$$

$$\delta\hat{x}^{n(r)\mu I} = \hat{\beta}^{pI}\partial_p \hat{x}^{n(r)\mu I}, \; \delta\hat{h}^I_{mn} = \hat{\beta}^{pI}\partial_p \hat{h}^I_{mn} + \partial_m \hat{\beta}^{pI}\hat{h}^I_{pn} + \partial_n \hat{\beta}^{pI}\hat{h}^I_{pm} \tag{2.60b}$$

$$\beta^{pI} \equiv \sum_u (-1)^{Iu} \hat{\beta}^{pu}. \tag{2.60c}$$

where I have relabelled the integration variables $t, \xi \to t_I, \xi_I$ separately in each term. In this form, one sees that the action is invariant — at least locally in the bulk — under the product of two diffeomorphism groups:

$$\xi_I^{m\prime} = \xi_I^{m\prime}(\{\xi_I^p\}), \quad m=0,1, \; I=0,1 \tag{2.61a}$$

$$\hat{x}^{n(r)\mu I\prime}(\{\xi_I^{p\prime}\}) = \hat{x}^{n(r)\mu I}(\{\xi_I^p\}) \tag{2.61b}$$

$$\hat{h}^{I\prime}_{mn}(\{\xi_I^{p\prime}\}) = \frac{\partial \xi_I^r}{\partial \xi_I^{m\prime}} \frac{\partial \xi_I^s}{\partial \xi_I^{n\prime}} \hat{h}^I_{rs}(\{\xi_I^p\}). \tag{2.61c}$$

Up to coupling at the branes then (see below), each of the two metrics $\hat{h}^I_{mn}$ transforms as an ordinary metric under its group.

In principle, one can use various finite symmetries of the action (2.60a) and invertibility of the definitions (2.59) to work out the form of these transformations in the original u-basis. Again up to coupling at the branes, consider the simple example of extended Weyl

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invariance:

$$\hat{h}^I_{mn} \to e^{-\hat{\sigma}_I}\hat{h}^I_{mn} : \tag{2.62a}$$

$$\hat{h}^{(u)}_{mn} \to \frac{1}{2}\sum_w (e^{-\hat{\sigma}_0} + (-1)^{u+w}e^{-\hat{\sigma}_1})\hat{h}^{(w)}_{mn} \tag{2.62b}$$

$$\tilde{h}^{mn}_{(u)} \to \tilde{h}^{mn}_{(u)}, \quad \hat{S} \to \hat{S}. \tag{2.62c}$$

Similarly, the following steps

$$h^I_{mn} = e^{-\hat{\phi}_I}\eta_{mn} : \tag{2.63a}$$

$$\hat{h}^{(u)}_{mn} = \frac{1}{2}\eta_{mn}(e^{-\hat{\phi}_0} + (-1)^u e^{-\hat{\phi}_1}) \tag{2.63b}$$

give the parametrization (2.51) quoted above for the extended metric in the conformal gauge.

I leave for another time and place the precise form of the finite $\mathbb{Z}_2$-diffeomorphisms in the u-basis. For this application in particular however, I emphasize that the decoupled form (2.60) of the bulk Lagrange density in the I-basis is quite *deceptive*, the complexity of the theory — and in particular its fractional modeing — being hidden in the unfamiliar boundary conditions

$$\sum_I (-1)^I \tilde{h}^{00}_I(0) = \sum_I (-1)^I \tilde{h}^{11}_I(0) = \sum_I \tilde{h}^{01}_I(0) = 0 \tag{2.64a}$$

$$\sum_I \dot{\hat{x}}^{n(r)\mu I}(0) = \sum_I (-1)^I \partial_\xi \hat{x}^{n(r)\mu I}(0) = 0 \tag{2.64b}$$

$$\tilde{h}^{01}_I(\pi) = 0 \tag{2.64c}$$

$$\dot{\hat{x}}^{n(r)\mu I}(\pi) + i\tan\left(\frac{n(r)\pi}{\rho(\sigma)}\right)\tilde{h}^{11}_I(\pi)\partial_\xi \hat{x}^{n(r)\mu I}(\pi) = 0 \tag{2.64d}$$

which couple the I-basis fields at the branes, even in the conformal gauge.

I finally mention that the basis change (2.59d) of the extended coordinates allows the extended action (2.58) of Nambu-type to be similarly expressed in terms of two ordinary untwisted Nambu actions [65], but the coupled coordinate boundary conditions at the branes are more intricate in this case.

3. General permutation gravity in the permutation orbifolds

3.1 Extended Polyakov hamiltonian

The (closed-string) WZW permutation orbifolds [27, 29, 31–33, 35, 36]

$$\frac{A_g(H)}{H}, \quad g = \oplus_{I=0}^{K-1}\mathbf{g}^I, \quad \mathbf{g}^I \simeq \mathbf{g}, \quad H = H(\text{perm}) \subset \text{Aut}(g) \tag{3.1}$$

have now been studied in considerable detail, where $H(\text{perm})$ is any permutation group which acts on K copies of $\mathbf{g}$ in g. We know in particular that the sectors $\{\sigma\}$ of these orbifolds are labelled by the conjugacy classes of $H(\text{perm})$, and each sector lives at central charge $\hat{c} = Kc_{\mathbf{g}}$ where $c_{\mathbf{g}}$ is the central charge of the affine-Sugawara construction [2, 3, 4, 7, 11] on $\mathbf{g}$. Moreover, each twisted sector ($\sigma \neq 0$) is governed by the following left-mover orbifold Virasoro algebra [35]

$$\left[\hat{L}_{\hat{j}j}\left(m+\frac{\hat{j}}{f_j(\sigma)}\right), \hat{L}_{\hat{l}l}\left(n+\frac{\hat{l}}{f_l(\sigma)}\right)\right] = \delta_{jl}\left\{\left(m-n+\frac{\hat{j}-\hat{l}}{f_j(\sigma)}\right)\hat{L}_{\hat{j}+\hat{l},j}\left(m+n+\frac{\hat{j}+\hat{l}}{f_j(\sigma)}\right)\right. \tag{3.2a}$$

$$\left. +\frac{c_{\mathbf{g}} f_j(\sigma)}{12}\left(m+\frac{\hat{j}}{f_j(\sigma)}\right)\left(\left(m+\frac{\hat{j}}{f_j(\sigma)}\right)^2-1\right)\delta_{m+u+\frac{\hat{j}+\hat{l}}{f_j(\sigma)},0}\right\}$$

$$\bar{\hat{j}}, \bar{\hat{l}} = 0, 1, \ldots, f_j(\sigma)-1, \quad \sum_j f_j(\sigma) = K \tag{3.2b}$$

as well as a commuting set of (rectified) right-mover copies $\{\hat{\bar{L}}^{\#}_{\hat{j}j}\}$. The physical Virasoro generators of each twisted sector

$$\hat{L}_\sigma(m) \equiv \sum_j \hat{L}_{0j}(m), \quad \hat{\bar{L}}_\sigma(m) \equiv \sum_j \hat{\bar{L}}^{\#}_{0j}(m) \tag{3.3a}$$

$$\hat{c} = \hat{\bar{c}} = c = Kc_{\mathbf{g}} \tag{3.3b}$$

are twisted affine-Sugawara constructions with the same central charges (3.3b) as the untwisted sector $\sigma = 0$.

The extended Virasoro algebra in eq. (3.2) is given in the standard [33–35, 37] cycle notation for the corresponding representative element $h_\sigma \in H(\text{perm})$, where j indexes the disjoint cycles of size $f_j(\sigma)$ and $\hat{j}$ indexes the position in the jth cycle. As examples, one has

$$\mathbb{Z}_\lambda : K = \lambda,\ f_j(\sigma) = \rho(\sigma),\ \bar{\hat{j}} = 0, \ldots, \rho(\sigma)-1, \tag{3.4a}$$

$$j = 0, \ldots, \frac{1}{\rho(\sigma)} - 1,\ \sigma = 0, \ldots, \rho(\sigma) - 1$$

$$\mathbb{Z}_\lambda,\ \lambda \text{ prime} : \rho(\sigma) = \lambda,\ \bar{\hat{j}} = 0, \ldots, \lambda - 1, \tag{3.4b}$$

$$j = 0,\ \sigma = 1, \ldots, \lambda - 1$$

$$S_N : K = N,\ f_j(\sigma) = \sigma_j,\ \sigma_{j+1} \leq \sigma_j, \tag{3.4c}$$

$$j = 0, \ldots, n(\vec{\sigma}) - 1,\ \sum_{j=0}^{n(\vec{\sigma})} \sigma_j = N.$$

For the special case $H = \mathbb{Z}_2$, the extended Virasoro algebra (3.2) reduces to the order-two orbifold Virasoro algebra (2.2a) studied above for the orientation orbifolds. More generally, the extended algebra (3.2) is semisimple, with one component for each cycle j in $h_\sigma \in H$.

For the classical development below, we will need the left- and right-mover extended stress tensors of each twisted sector [35]

$$\hat{\theta}_{\hat{j}j}(\xi,t) = \frac{1}{2\pi}\sum_{m\in\mathbb{Z}} \hat{L}_{\hat{j}j}\left(m+\frac{\hat{j}}{f_j(\sigma)}\right)e^{-i\left(m+\frac{\hat{j}}{f_j(\sigma)}\right)(t+\xi)} \tag{3.5a}$$

$$\hat{\bar{\theta}}_{\hat{j}j}(\xi,t) = \frac{1}{2\pi}\sum_{m\in\mathbb{Z}} \hat{\bar{L}}^{\#}_{-\hat{j},j}(m-\tfrac{\hat{j}}{f_j(\sigma)})e^{-i\left(m-\frac{\hat{j}}{f_j(\sigma)}(t-\xi)\right)} \tag{3.5b}$$

$$\hat{\theta}_{\hat{j}j}(\xi+2\pi,t) = e^{-2\pi i\frac{\hat{j}}{f_j(\sigma)}}\hat{\theta}_{\hat{j}j}(\xi,t),\;\; \hat{\bar{\theta}}_{\hat{j}j}(\xi+2\pi,t) = e^{-2\pi i\frac{\hat{j}}{f_j(\sigma)}}\hat{\bar{\theta}}_{\hat{j}j}(\xi,t) \tag{3.5c}$$

$$\{\hat{\theta}_{\hat{j}j}(\xi,t),\hat{\theta}_{\hat{l}l}(\eta,t)\} = i\delta_{jl}(\partial_\xi-\partial_\eta)(\hat{\theta}_{\hat{j}+\hat{l}}(\eta)\delta_{\frac{\hat{j}}{f_j(\sigma)}}(\xi-\eta)) \tag{3.5d}$$

$$\{\hat{\bar{\theta}}_{\hat{j}j}(\xi,t),\hat{\bar{\theta}}_{\hat{l}l}(\eta,t)\} = -i\delta_{jl}(\partial_\xi-\partial_\eta)(\hat{\bar{\theta}}_{\hat{j}+\hat{l}}(\eta)\delta_{\frac{\hat{j}}{f_j(\sigma)}}(\xi-\eta)) \tag{3.5e}$$

$$\{\hat{\theta}_{\hat{j}j}(\xi,t),\hat{\bar{\theta}}_{\hat{l}l}(\eta,t)\} = 0 \tag{3.5f}$$

$$\delta_{\frac{\hat{j}}{f_j(\sigma)}}(\xi-\eta) = \frac{1}{2\pi}\sum_{m\in\mathbb{Z}} e^{-i\left(m+\frac{\hat{j}}{f_j(\sigma)}\right)(\xi-\eta)} = \delta_{-\frac{\hat{j}}{f_j(\sigma)}}(\eta-\xi). \tag{3.5g}$$

whose properties follow again from the classical analogue of eq. (3.2b). The barred brackets here follow from the unbarred because $\hat{\bar{\theta}}_{-\hat{j},j}$ has the same brackets as $\hat{\theta}_{\hat{j}j}$ with $\xi \to -\xi$.

Then following the development above for the orientation orbifolds, I define the (monodromy-invariant) extended Hamiltonian and gauge generator for each sector σ of each permutation orbifold

$$\hat{H}_\sigma \equiv \int_0^\pi d\xi \sum_j \sum_{\hat{j}=0}^{f_j(\sigma)-1} (\hat{v}^{\hat{j}j}\hat{\theta}_{\hat{j}j} + \hat{\bar{v}}^{\hat{j}j}\hat{\bar{\theta}}_{\hat{j}j}) \tag{3.6a}$$

$$\equiv \int_0^{2\pi} d\xi (\hat{v}^{\hat{j}j}\hat{\theta}_{\hat{j}j} + \hat{\bar{v}}^{\hat{j}j}\hat{\bar{\theta}}_{\hat{j}j}) \tag{3.6b}$$

$$\hat{G}_\sigma \equiv \int_0^{2\pi} d\xi (\hat{\epsilon}^{\hat{j}j}\hat{\theta}_{\hat{j}j} + \hat{\bar{\epsilon}}^{\hat{j}j}\hat{\bar{\theta}}_{\hat{j}j}) \tag{3.6c}$$

$$\hat{\mathcal{O}}^{\hat{j}j}(\xi+2\pi) = e^{2\pi i\frac{\hat{j}}{f_j(\sigma)}}\hat{\mathcal{O}}^{\hat{j}j}(\xi), \quad \hat{\mathcal{O}} = \{\hat{v},\hat{\bar{v}},\hat{\epsilon},\hat{\bar{\epsilon}}\} \tag{3.6d}$$

$$\dot{\hat{A}} \equiv i\{\hat{H}_\sigma,\hat{A}\}, \qquad \delta\hat{A} \equiv i\{\hat{G}_\sigma,\hat{A}\} \tag{3.6e}$$

$$\delta\hat{v}^{\hat{j}j} \equiv \sum_{\hat{l}=0}^{f_j(\sigma)-1} \hat{\epsilon}^{\hat{j}-\hat{l},j}\overleftrightarrow{\partial_\xi}\,\hat{v}^{\hat{l}j}, \quad \delta\hat{\bar{v}}^{\hat{j}j} \equiv \sum_{\hat{l}=0}^{f_j(\sigma)-1} \hat{\bar{v}}^{\hat{j}-\hat{l},j}\overleftrightarrow{\partial_\xi}\,\hat{\bar{\epsilon}}^{\hat{l}j} \tag{3.6f}$$

where $\hat{v},\hat{\bar{v}}$ are the multipliers and $\hat{\epsilon},\hat{\bar{\epsilon}}$ are the (time-independent) extended gauge parameters. This gives in particular the properties of the gauge-variant stress tensors and multipliers

$$\delta\hat{\theta}_{\hat{j}j} = \sum_{\hat{l}=0}^{f_j(\sigma)-1}\left[\partial_\xi(\hat{\theta}_{\hat{j}+\hat{l},j}\hat{\epsilon}^{\hat{l}j}) + \hat{\theta}_{\hat{j}+\hat{l},j}\partial_\xi\hat{\epsilon}^{\hat{l}j}\right] \tag{3.7a}$$

$$\delta\hat{\bar{\theta}}_{\hat{j}j} = -\sum_{\hat{l}=0}^{f_j(\sigma)-1}\left[\partial_\xi(\hat{\bar{\theta}}_{\hat{j}+\hat{l},j}\hat{\bar{\epsilon}}^{\hat{l}j}) + \hat{\bar{\theta}}_{\hat{j}+\hat{l},j}\partial_\xi\hat{\bar{\epsilon}}^{\hat{l}j}\right] \tag{3.7b}$$

$$\dot{\hat{\theta}}_{\hat{j}j} = \sum_{\hat{l}=0}^{f_j(\sigma)-1}\left[\partial_\xi(\hat{\theta}_{\hat{j}+\hat{l},j}\hat{v}^{\hat{l}j}) + \hat{\theta}_{\hat{j}+\hat{l},j}\partial_\xi\hat{v}^{\hat{l}j}\right] \tag{3.7c}$$

$$\dot{\hat{\bar{\theta}}}_{\hat{j}j} = -\sum_{\hat{l}=0}^{f_j(\sigma)-1}\left[\partial_\xi(\hat{\bar{\theta}}_{\hat{j}+\hat{l}}\hat{\bar{v}}^{\hat{l}j}) + \hat{\bar{\theta}}_{\hat{j}+\hat{l},j}\partial_\xi\hat{\bar{v}}^{\hat{l}j}\right] \tag{3.7d}$$

$$\dot{\hat{v}}^{\hat{j}j} = -\sum_{\hat{l}=0}^{f_j(\sigma)-1}\hat{v}^{\hat{j}-\hat{l},j}\overleftrightarrow{\partial}_\xi\,\hat{v}^{\hat{l}j}, \quad \dot{\hat{\bar{v}}}^{\hat{j}j} = \sum_{\hat{l}=0}^{f_j(\sigma)-1}\hat{\bar{v}}^{\hat{j}-\hat{l},j}\overleftrightarrow{\partial}_\xi\,\hat{\bar{v}}^{\hat{l}j}. \tag{3.7e}$$

where (3.7e) follows from (3.7c), (3.7d) and the requirement that $\dot{\hat{H}}_\sigma = 0$. Then it is easily checked that the extended Hamiltonian is gauge-invariant $\delta\hat{H}_\sigma = 0$ under the extended gauge group of the orbifold Virasoro algebra (3.2a). Note also that the monodromies, dynamics and gauge transformations in eqs. (3.6), (3.7) do not mix the cycles $\{j\}$.

The corresponding multiplier equations of motion are the extended Virasoro (Polyakov) constraints

$$\hat{\theta}_{\hat{j}j} = \hat{\bar{\theta}}_{\hat{j}j} = 0, \quad \bar{\hat{j}} = 0,1,\ldots,f_j(\sigma)-1, \quad \sum_j f_j(\sigma) = K \tag{3.8a}$$

$$\leftrightarrow \{\hat{L}_{\hat{j}j}\left(m + \frac{\hat{j}}{f_j(\sigma)}\right) = \hat{\bar{L}}^{\#}_{\hat{j}j}\left(m + \frac{\hat{j}}{f_j(\sigma)}\right) = 0\} \tag{3.8b}$$

so we will not be surprised to find these constraints again in the action formulation below. I emphasize that the number of extended Virasoro constraints (which is also the number of extended gauge degrees of freedom) is

$$N_* = 2\sum_j f_j(\sigma) = 2K. \tag{3.9}$$

This counting holds in all sectors of each permutation orbifold, including the untwisted sector $\sigma = 0$ where K is the number of copies of $\mathbf{g}$ in g.

Among possible Hamiltonian gauge choices, I mention the following: In the (completely-fixed) conformal gauge, the extended Hamiltonian reduces to the CFT Hamiltonian of each sector

$$\hat{v}^{\hat{j}j} = \hat{\bar{v}}^{\hat{j}j} = \delta_{\hat{j},0 \bmod f_j(\sigma)} \tag{3.10a}$$

$$\hat{H}_\sigma = \int_0^{2\pi} d\xi \sum_j (\hat{\theta}_{0j} + \hat{\bar{\theta}}_{0j}) = \hat{L}_\sigma(0) + \hat{\bar{L}}_\sigma(0) \tag{3.10b}$$

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where the extended stress tensors of each twisted closed-string CFT are given in eq. (3.5). Other (partially-fixed) gauges of interest include the extended Polyakov gauge

$$\hat{v}^{\hat{j}j} = \hat{v}^{j}\delta_{\hat{j},0 \bmod f_j(\sigma)}, \quad \hat{\bar{v}}^{\hat{j}j} = \hat{\bar{v}}^{j}\delta_{\hat{j},0 \bmod f_j(\sigma)} \tag{3.11}$$

which corresponds in fact to choosing a distinct (ordinary) Polyakov metric for each disjoint cycle j, and the Polyakov gauge

$$\hat{v}^{\hat{j}j} = \hat{v}\delta_{\hat{j},0 \bmod f_j(\sigma)}, \quad \hat{\bar{v}}^{\hat{j}j} = \hat{\bar{v}}\delta_{\hat{j},0 \bmod f_j(\sigma)} \tag{3.12a}$$

$$\hat{H}_\sigma = \int_0^{2\pi} d\xi(\hat{v}\hat{\theta} + \hat{\bar{v}}\hat{\bar{\theta}}) \tag{3.12b}$$

$$\hat{\theta} \equiv \sum_j \hat{\theta}_{0j}, \quad \hat{\bar{\theta}} \equiv \sum_j \hat{\bar{\theta}}_{0j} \tag{3.12c}$$

where $\hat{\theta}$ and $\hat{\bar{\theta}}$ are the physical stress tensors (see eq. (3.3)) of sector σ. This is the ordinary Polyakov Hamiltonian of the sector, in which the same Polyakov metric is chosen for all the cycles.

To go further, we need the explicit phase-space formulation of the extended stress tensors in each twisted sector of the permutation orbifolds.[6] This data is given for the WZW permutation orbifolds in refs. [35,37], but I limit the discussion here to the ($B = \hat{B} = 0$) free-bosonic permutation orbifolds

$$\frac{\mathrm{U}(1)^{Kd}}{H(\mathrm{perm})}, \quad \mathrm{U}(1)^{Kd} \equiv \mathrm{U}(1)^d \times \mathrm{U}(1)^d \times \ldots (K \text{ times}) \tag{3.13}$$

all of whose sectors have central charge $\hat{c} = c = Kd$.

The free-bosonic permutation orbifolds have also been worked out in refs. [33, 35, 37] as a special case on abelian g, and we may read off from these references (at $k = 1$ for simplicity):

$$\hat{\theta}_{\hat{j}j} = \frac{1}{4\pi}\frac{G^{ab}}{f_j(\sigma)} \sum_{\hat{l}=0}^{f_j(\sigma)-1} \hat{J}_{\hat{l}aj}\hat{J}_{\hat{j}-\hat{l},bj} \tag{3.14a}$$

$$\hat{\bar{\theta}}_{\hat{j}j} = \frac{1}{4\pi}\frac{G^{ab}}{f_j(\sigma)} \sum_{\hat{l}=0}^{f_j(\sigma)-1} \hat{\bar{J}}_{\hat{l}aj}\hat{\bar{J}}_{\hat{j}-\hat{l},bj} \tag{3.14b}$$

$$\hat{J}_{\hat{j}aj}(\xi + 2\pi) = e^{-2\pi i \frac{\hat{j}}{f_j(\sigma)}}\hat{J}_{\hat{j}aj}(\xi), \quad \hat{\bar{J}}_{\hat{j}aj}(\xi + 2\pi) = e^{-2\pi i \frac{\hat{j}}{f_j(\sigma)}}\hat{\bar{J}}_{\hat{j}aj}(\xi) \tag{3.14c}$$

$$\{\hat{J}_{\hat{j}aj}(\xi), \hat{J}_{\hat{l}bl}(\eta)\} = -\{\hat{\bar{J}}_{\hat{j}aj}(\xi), \hat{\bar{J}}_{\hat{l}bl}(\eta)\} \tag{3.14d}$$

$$= \delta_{jl}\delta_{\hat{j}+\hat{l},0 \bmod f_j(\sigma)} f_j(\sigma) G_{ab}\partial_\xi \delta_{\frac{\hat{j}}{f_j(\sigma)}}(\xi - \eta) \tag{3.14e}$$

[6] Although WZW was used as an illustration above, the orbifold Virasoro algebra (3.2a) and the extended Hamiltonian system (3.6) hold for general permutation orbifolds, including sigma-model permutation orbifolds (see ref. [37]).

$$\hat{J}_{\hat{j}aj} = 2\pi\hat{p}_{\hat{j}aj} + \frac{f_j(\sigma)}{2}G_{ab}\partial_\xi\hat{x}^{-\hat{j},bj} \tag{3.14f}$$

$$\hat{\bar{J}}_{\hat{j}aj} = -2\pi\hat{p}_{\hat{j}aj} + \frac{f_j(\sigma)}{2}G_{ab}\partial_\xi\hat{x}^{-\hat{j},bj} \tag{3.14g}$$

$$\hat{x}^{\hat{j}aj}(\xi+2\pi) = e^{2\pi i\frac{\hat{j}}{f_j(\sigma)}}\hat{x}^{\hat{j}aj}, \quad \hat{p}_{\hat{j}aj}(\xi+2\pi) = e^{-2\pi i\frac{\hat{j}}{f_j(\sigma)}}\hat{p}_{\hat{j}aj}(\xi) \tag{3.14h}$$

$$\{\hat{p}_{jaj}(\xi), \hat{x}^{\hat{l}bl}(\eta)\} = -i\delta_j^l\delta_a^b\delta_{\hat{j}-\hat{l},0 \text{ mod } f_j(\sigma)}\delta_{\frac{\hat{j}}{f_j(\sigma)}}(\xi-\eta). \tag{3.14i}$$

Here $\{\hat{x}\}$ and $\{\hat{p}\}$ are the twisted coordinates and momenta (there are Kd of each) in twisted sector σ of each free-bosonic permutation orbifold, and each twisted current algebra in (3.14d,e) is called an abelian orbifold affine algebra [27].

In further detail, the quantity G_{ab} (and its inverse G^{ab}) in (3.14) is the untwisted tangent-space metric for each closed string copy $\mathrm{U}(1)^d$ in the symmetric sector $\mathrm{U}(1)^{Kd}$, where the untwisted current algebras, permutations and H-eigenvalue problem read:

$$[J_{aI}(m), J_{bJ}(n)] = [\bar{J}_{aI}(m), \bar{J}_{bJ}(n)] = m\delta_{IJ}G_{ab}\delta_{m+n,0} \tag{3.15a}$$

$$J_{aI}(m)' = \omega(\sigma)_I{}^J J_{aJ}(m), \quad \hat{\bar{J}}_{aI}(m)' = \omega(\sigma)_I{}^J \bar{J}_{aJ}(m) \tag{3.15b}$$

$$\omega(\sigma) \in H(\text{perm}) \tag{3.15c}$$

$$\omega(\sigma)_I{}^J U^\dagger(\sigma)_J{}^{\hat{j}j} = U^\dagger(\sigma)_I{}^{\hat{j}j} e^{-2\pi i\frac{\hat{j}}{f_j(\sigma)}}, \tag{3.15d}$$

$$\bar{\hat{j}} = 0, \ldots, f_j(\sigma)-1, \quad \sum_j f_j(\sigma) = K \tag{3.15e}$$

$$m,n \in \mathbb{Z}, \;\; a,b = 1,\ldots,d, \;\; I,J = 0,\ldots,K-1. \tag{3.15f}$$

This is the same G_{ab} introduced for the untwisted open string in section 2, and the same value

$$\frac{\mathrm{U}(1)^{26K}}{H(\text{perm})}: \quad G_{ab} = G^{ab} = \begin{pmatrix} -1 & 0 \\ 0 & 1\!\!1 \end{pmatrix}, \;\; a,b = 0,1,\ldots,25 \tag{3.16a}$$

$$\hat{c} = c = 26K \tag{3.16b}$$

can be chosen at any point in the discussion below to obtain the results for the critical permutation orbifolds in Minkowski space.

It is now straightforward to work out the Hamiltonian equation of motion of the twisted coordinates

$$\dot{\hat{x}}^{\hat{j}aj} = \frac{G^{ab}}{f_j(\sigma)}\sum_{\hat{l}=0}^{f_j(\sigma)-1}\left(\hat{v}^{\hat{l}j}\hat{J}_{\hat{l}-\hat{j},bj} - \hat{\bar{v}}^{\hat{l}j}\hat{\bar{J}}_{\hat{l}-\hat{j},bj}\right) \tag{3.17}$$

and use the phase-space realization (3.14f,g) of the currents to construct the extended action

$$\hat{S}_\sigma = \int dt \int_0^{2\pi} d\xi \sum_{j,\hat{j}}\left(\dot{\hat{x}}^{\hat{j}aj}\hat{p}_{\hat{j}aj} - \hat{v}^{\hat{j}j}\hat{\theta}_{\hat{j}j} - \hat{\bar{v}}^{\hat{j}j}\hat{\bar{\theta}}_{\hat{j}j}\right) \tag{3.18}$$

in each sector σ of all the permutation orbifolds. Again following refs. [63, 61], we know that each extended action is invariant $\delta\hat{S}_\sigma = 0$ under the following full time-dependent gauge transformations

$$\delta \hat{v}^{\hat{j}j} = \dot{\hat{\epsilon}}^{\hat{j}j} + \sum_{\hat{l}=0}^{f_j(\sigma)-1} \hat{\epsilon}^{\hat{j}-\hat{l},j} \overleftrightarrow{\partial_\xi} \hat{v}^{\hat{l}j} \tag{3.19a}$$

$$\delta \hat{\bar{v}}^{\hat{j}j} = \dot{\hat{\bar{\epsilon}}}^{\hat{j}j} + \sum_{\hat{l}=0}^{f_j(\sigma)-1} \hat{\bar{v}}^{\hat{j}-\hat{l},j} \overleftrightarrow{\partial_\xi} \hat{\bar{\epsilon}}^{\hat{l}j} \tag{3.19b}$$

and $\delta \hat{A}$ in eq. (3.6e) for the matter fields. These transformations define the phase-space form of the extended, twisted diffeomorphisms of general permutation gravity.

3.2 Extended Polyakov actions

In order to keep track of the branes, I followed the transition to coordinate space carefully for the open-string sectors of the orientation orbifolds. Such detail is unnecessary for the closed-string sectors of the permutation orbifolds because boundary conditions are now replaced by *monodromies*, and these are simple to track for all fields. Indeed, consulting eqs. (3.5c), (3.6d) and (3.14h), we see a $+$ or $-$ phase under $\xi \to \xi + 2\pi$ for each up or down index $\hat{j}$ respectively.

I therefore present only the final coordinate-space form of $\hat{S}_\sigma$ in eq. (3.18), which I will call the *general extended action* of Polyakov-type:

$$\hat{\mathcal{L}}^{\sigma}_{\hat{j}j} \equiv \frac{1}{8\pi} \sum_{\hat{l},\hat{m}=0}^{f_j(\sigma)-1} \tilde{h}^{mn}_{(\hat{j}+\hat{l}+\hat{m})j} \partial_m \hat{x}^{\hat{l}aj} (f_j(\sigma) G_{ab}) \partial_n \hat{x}^{\hat{m}bj} \tag{3.20a}$$

$$\hat{S}_\sigma = \int dt \int_0^{2\pi} d\xi \sum_j \hat{\mathcal{L}}^{\sigma}_{0j} \tag{3.20b}$$

$$= \frac{1}{8\pi} \int dt \int_0^{2\pi} d\xi \sum_{j\hat{l}\hat{m}} \tilde{h}^{mn}_{(\hat{l}+\hat{m})j} \partial_m \hat{x}^{\hat{l}aj} f_j(\sigma) G_{ab} \partial_n \hat{x}^{\hat{m}bj} \tag{3.20c}$$

$$\hat{x}^{\hat{j}aj}(\xi + 2\pi, t) = e^{2\pi i \frac{\hat{j}}{f_j(\sigma)}} \hat{x}^{\hat{j}aj}(\xi, t) \tag{3.20d}$$

$$\tilde{h}^{mn}_{(\hat{j})j}(\xi + 2\pi, t) = e^{-2\pi i \frac{\hat{j}}{f_j(\sigma)}} \tilde{h}^{mn}_{(\hat{j})j}(\xi, t) \tag{3.20e}$$

$$\hat{\mathcal{L}}^{\sigma}_{\hat{j}j}(\xi + 2\pi, t) = e^{-2\pi i \frac{\hat{j}}{f_j(\sigma)}} \hat{\mathcal{L}}^{\sigma}_{\hat{j}j}(\xi, t). \tag{3.20f}$$

For each sector σ of each permutation orbifold the general extended action density is cycle-separable and monodromy-invariant. The general permutation-twisted gravitational structure

$$\tilde{h}^{mn}_{(\hat{j})j}(\xi, t) = \tilde{h}^{nm}_{(\hat{j})j}(\xi, t) \tag{3.21a}$$

$$m, n \in (0, 1), \quad \bar{\hat{j}} = 0, \ldots, f_j(\sigma) - 1, \quad \sum_j f_j(\sigma) = K \tag{3.21b}$$

collects all dependence on the phase-space multipliers $\{\hat{v}, \hat{\bar{v}}\}$, and hence possesses $N_* = 2K$ independent degrees of freedom. I will again call this structure the *inverse extended metric* *density* of general permutation gravity, though we shall see momentarily that it is in fact a set of inverse extended metric densities, one for each cycle j of each representative element $h_\sigma \in H(\text{perm})$.

The coordinate-space form of the infinitesimal twisted diffeomorphisms (3.6e), (3.19a) of general permutation gravity is

$$\delta\hat{x}^{\hat{j}aj} = \sum_{\hat{l}=0}^{f_j(\sigma)-1} \hat{\beta}^{p\hat{l}j}\partial_p\hat{x}^{\hat{j}-\hat{l},aj} \tag{3.22a}$$

$$\hat{\beta}^{m\hat{j}j}(\xi+2\pi,t) = e^{2\pi i\frac{\hat{j}}{f_j(\sigma)}}\hat{\beta}^{m\hat{j}j}(\xi,t) \tag{3.22b}$$

$$\delta\tilde{h}^{mn}_{(\hat{j})j} = \sum_{\hat{l}=0}^{f_j(\sigma)-1}\Big(\partial_p(\hat{\beta}^{p\hat{l}j}\tilde{h}^{mn}_{(\hat{j}+\hat{l})j}) - (\partial_p\hat{\beta}^{m\hat{l}j})\tilde{h}^{pn}_{(\hat{j}+\hat{l})j} - (\partial_p\hat{\beta}^{n\hat{l}j}\tilde{h}^{pm}_{(\hat{j}+\hat{l})j})\Big) \tag{3.22c}$$

$$\delta\hat{\mathcal{L}}^\sigma_{0j} = \partial_p\left(\sum_{\hat{j}=0}^{f_j(\sigma)-1}\hat{\beta}^{p\hat{j}j}\hat{\mathcal{L}}^\sigma_{\hat{j}j}\right) \tag{3.22d}$$

where $\{\hat{\beta}\}$ are the $N_* = 2K$ independent coordinate-space extended diffeomorphism parameters. It follows that $\delta\hat{S}_\sigma = 0$, so that the general extended action has the expected invariance under the extended diffeomorphisms. Moreover, we see that the extended diffeomorphisms do not mix the different cycle-components $\{j\}$ of the extended inverse metric density.

I close this subsection with the twisted-coordinate equations of motion

$$\partial_m\left(\sum_{\hat{l}=0}^{f_j(\sigma)-1}\tilde{h}^{mn}_{(\hat{j}+\hat{l})j}\partial_n\hat{x}^{\hat{l}aj}\right) = 0 \tag{3.23a}$$

$$\bar{\hat{j}} = 0,1,\ldots,f_j(\sigma)-1, \qquad \sum_j f_j(\sigma) = K, \quad a = 1,\ldots,d \tag{3.23b}$$

which follow by arbitrary variation $\delta\hat{x}$ of the general extended action, and two simple checks of our results up to this point: First, for the single twisted sector of the $\mathbb{Z}_2$-permutation orbifold, one sees that our results reduce locally to the *same* $\mathbb{Z}_2$-permutation gravity found in the (open-string) orientation orbifold sectors. In particular, the extended inverse metric density in this case takes the form $\tilde{h}^{mn}_{(\hat{j})0}$, $\bar{\hat{j}} = 0,1$ because the non-trivial element of $\mathbb{Z}_2$ is a single cycle. This makes sense because the local gravitational structure is governed by an order-two orbifold Virasoro algebra in both cases. Second, the general extended action (3.20) reduces in the case of the (completely-fixed) conformal gauge (3.10) to the known conformal-field-theoretic action of each sector of the free-bosonic permutation orbifolds [37]

$$\tilde{h}^{mn}_{(\hat{j})j} = \eta^{mn}\delta_{\hat{j},0 \bmod f_j(\sigma)}, \quad \eta = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \tag{3.24a}$$

$$\hat{S}_\sigma = \frac{1}{8\pi}\int dt\int_0^{2\pi} d\xi\eta^{mn}\sum_j f_j(\sigma)\sum_{\hat{j}=0}^{f_j(\sigma)-1}\partial_m\hat{x}^{\hat{j}aj}G_{ab}\partial_n\hat{x}^{-\hat{j},bj} \tag{3.24b}$$

where η is the flat-world sheet metric. The coordinate-monodromies given in eq. (3.20d) are of course independent of gauge choice, while the monodromy (3.20e) of the inverse extended metric density is now trivial because its conformal-gauge support is only at $\bar{\hat{j}} = 0$.

3.3 The twisted permutation gravities

I turn next to the identification of the *extended world-sheet metric* of each twisted permutation gravity:

$$\hat{h}_{mn}^{(\hat{j})j} = \hat{h}_{nm}^{(\hat{j})j}, \quad \bar{\hat{j}} = 0, \ldots, f_j(\sigma) - 1, \ \sum_j f_j(\sigma) = K, \ m, n \in (0,1). \tag{3.25}$$

Generalizing our discussion of $\mathbb{Z}_2$-permutation gravity in section 2, the extended metric (and its inverse $\hat{h}^{mn}_{(\hat{j})j}$) can be identified by the following decomposition of the extended inverse metric density:

$$\tilde{h}^{mn}_{(\hat{j})j} = \sum_{\hat{l}=0}^{f_j(\sigma)-1} \hat{h}^{mn}_{(\hat{j}+\hat{l})} \hat{H}^{(\hat{l})j}, \quad \hat{h}^{mn}_{(\hat{j})j} = \hat{h}^{nm}_{(\hat{j})j} \tag{3.26a}$$

$$\sum_{\hat{k}=0}^{f_j(\sigma)-1} \hat{h}^{mp}_{(\hat{j}+\hat{k})j} \hat{h}_{pn}^{(\hat{k}+\hat{l})j} = \delta_n^m \delta_{\hat{j}-\hat{l},0 \bmod f_j(\sigma)}, \ \forall\, j \tag{3.26b}$$

$$\hat{H}^{(\hat{j})j} = \frac{1}{f_j(\sigma)} \sum_{J=0}^{f_j(\sigma)-1} e^{-2\pi i \frac{\hat{j}J}{f_j(\sigma)}} \left(-\det \left(\sum_{\hat{l}=0}^{f_j(\sigma)-1} e^{2\pi i \frac{J\hat{l}}{f_j(\sigma)}} \hat{h}_{mn}^{(\hat{l})j} \right) \right)^{\frac{1}{2}} \tag{3.26c}$$

As above, the determinant in eq. (3.26c) operates only on the 2×2 world-sheet indices m, n. It is then straightforward to check that the following permutation-twisted diffeomorphisms and monodromies

$$\delta\hat{h}_{mn}^{(\hat{j})j} = \sum_{\hat{l}=0}^{f_j(\sigma)-1} \left(\hat{\beta}^{p\hat{l}j} \partial_p \hat{h}_{mn}^{(\hat{j}-\hat{l})j} + \partial_m \hat{\beta}^{p\hat{l}j} \hat{h}_{pn}^{(\hat{j}-\hat{l})j} + \partial_n \hat{\beta}^{p\hat{l}j} \hat{h}_{pm}^{(\hat{j}-\hat{l})j} \right) \tag{3.27a}$$

$$\delta\hat{h}^{mn}_{(\hat{j})j} = \sum_{\hat{l}=0}^{f_j(\sigma)-1} \left(\hat{\beta}^{p\hat{l}j} \partial_p \hat{h}^{mn}_{(\hat{j}+\hat{l})j} - \partial_p \hat{\beta}^{m\hat{l}j} \hat{h}^{pn}_{(\hat{j}+\hat{l})j} - \partial_p \hat{\beta}^{n\hat{l}j} \hat{h}^{pm}_{(\hat{j}+\hat{l})j} \right) \tag{3.27b}$$

$$\delta\hat{H}^{(\hat{j})j} = \partial_p \left(\sum_{\hat{l}=0}^{f_j(\sigma)-1} \beta^{p\hat{l}j} \hat{H}^{(\hat{j}-\hat{l})j} \right) \tag{3.27c}$$

$$\hat{h}_{mn}^{(\hat{j})j}(\xi + 2\pi, t) = e^{2\pi i \frac{\hat{j}}{f_j(\sigma)}} \hat{h}_{mn}^{(\hat{j})j}(\xi, t) \tag{3.27d}$$

$$\hat{h}^{mn}_{(\hat{j})j}(\xi + 2\pi, t) = e^{-2\pi i \frac{\hat{j}}{f_j(\sigma)}} \hat{h}^{mn}_{(\hat{j})j}(\xi, t) \tag{3.27e}$$

$$\hat{H}^{(\hat{j})j}(\xi + 2\pi, t) = e^{2\pi i \frac{\hat{j}}{f_j(\sigma)}} \hat{H}^{(\hat{j})j}(\xi, t) \tag{3.27f}$$

are consistent and reproduce the corresponding transformations in (3.20e), (3.22c) of the extended inverse metric density. We may take the transformations (3.27c), (3.27f) as defining a set of extended scalar densities, one for each cycle j of h_σ, although only the $\bar{\hat{j}} = 0$

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component of each has trivial monodromy. For the case $H(\text{perm}) = \mathbb{Z}_2$ the decomposition (3.26) and the twisted diffeomorphisms in (3.27) reduce to the $\mathbb{Z}_2$-gravitational results of Subsection 2.7.

More generally, these results describe a distinct, twisted permutation gravity in each twisted sector of each permutation orbifold. Correspondingly, the permutation gravities are in $1-1$ correspondence with the conjugacy classes of any permutation group $H(\text{perm})$. Although I have worked out the invariant actions only for the free-bosonic orbifolds

$$\hat{S}_\sigma = \frac{1}{8\pi}\int dt \int_0^{2\pi} d\xi \sum_j \sum_{\hat{\jmath},\hat{l},\hat{k}=0}^{f_j(\sigma)-1} \hat{h}^{mn}_{(\hat{\jmath}+\hat{l}+\hat{k})j} \hat{H}^{(\hat{\jmath})j} \partial_m \hat{x}^{\hat{l}aj} f_j(\sigma) G_{ab} \partial_n \hat{x}^{\hat{k}bj} \tag{3.28}$$

the extended Hamiltonian formulation of Subsection 3.1 tells us that the *same* P-gravitational structures will also appear in the extended actions of permutation orbifolds of general WZW and sigma models.

Note that the extended world-sheet metric $\hat{h}^{(\hat{\jmath})j}_{mn}$ of each twisted sector σ has

$$N'_* = 3K \tag{3.29}$$

independent components while there are only $N_* = 2K$ extended gauge degrees of freedom in each permutation gravity. Then the extended metric includes

$$N'_* - N_* = K \tag{3.30}$$

twisted Weyl degrees of freedom, which cannot be gauged away and which do not appear in the extended inverse metric density $\tilde{h}^{mn}_{(\hat{\jmath})j}$. This counting holds in all sectors of the permutation orbifolds, including the untwisted sector $\sigma = 0$ where we have K copies of the ordinary Polyakov metric in the world-sheet description of $\mathrm{U}(1)^{Kd}$. Indeed, I will argue in the following subsection that the extended metric of twisted sector σ can be parametrized in the (completely-fixed) conformal gauge as

$$\hat{h}^{(\hat{\jmath})j}_{mn} = \eta_{mn}\hat{H}^{(\hat{\jmath})j} \tag{3.31a}$$

$$\hat{H}^{(\hat{\jmath})j} = \frac{1}{f_j(\sigma)} \sum_{J=0}^{f_j(\sigma)-1} \exp\left(-2\pi i \frac{\hat{\jmath}J}{f_j(\sigma)} - \sum_{\hat{l}=0}^{f_j(\sigma)-1} e^{-2\pi i \frac{J\hat{l}}{f_j(\sigma)}} \hat{\phi}_{(\hat{\jmath})j}\right) \tag{3.31b}$$

$$\hat{h}^{mn}_{(\hat{\jmath})j} = \eta^{mn}\frac{1}{f_j(\sigma)} \sum_{J=0}^{f_j(\sigma)-1} \exp\left(2\pi i \frac{\hat{\jmath}J}{f_j(\sigma)} + \sum_{\hat{l}=0}^{f_j(\sigma)-1} e^{-2\pi i \frac{J\hat{l}}{f_j(\sigma)}} \hat{\phi}_{(\hat{\jmath})j}\right) \tag{3.31c}$$

$$\tilde{h}^{mn}_{(\hat{\jmath})j} = \eta^{mn}\delta_{\hat{\jmath},0 \bmod f_j(\sigma)} \tag{3.31d}$$

$$\hat{\phi}_{(\hat{\jmath})j}(\xi+2\pi, t) = e^{-2\pi i \frac{\hat{\jmath}}{f_j(\sigma)}} \hat{\phi}_{(\hat{\jmath})j}(\xi, t), \qquad \bar{\hat{\jmath}} = 0, \ldots, f_j(\sigma)-1, \ \sum_j f_j(\sigma) = K \tag{3.31e}$$

where η is the flat world-sheet metric and $\{\hat{\phi}\}$ are the K twisted Weyl fields.

As a final topic in this subsection, I consider the P-gravitational equations of motion, which are obtained by arbitrary variation $\delta\hat{h}^{(\hat{j})j}_{mn}$ of the extended metric in the extended action (3.28). After some algebra, the result is

$$\hat{\theta}^{\hat{j}j}_{mn} \equiv \frac{1}{2}\left(\hat{\mathcal{L}}^{\hat{j}j}_{mn} - \frac{1}{2}\sum_{\hat{l},\hat{m}=0}^{f_j(\sigma)-1} \hat{h}^{(\hat{l})j}_{mn}\hat{h}^{pq}_{(\hat{m})j}\hat{\mathcal{L}}^{\hat{m}-\hat{l}-\hat{j},j}_{pq}\right) \tag{3.32a}$$

$$\hat{\mathcal{L}}^{\hat{j}j}_{mn} \equiv \frac{1}{8\pi}\sum_{\hat{l}=0}^{f_j(\sigma)-1} \partial_m\hat{x}^{\hat{l}aj} f_j(\sigma)G_{ab}\partial_n\hat{x}^{\hat{j}-\hat{l},bj} \tag{3.32b}$$

$$\sum_{\hat{l}=0}^{f_j(\sigma)-1} \hat{h}^{mn}_{(\hat{j}+\hat{l})j}\hat{\theta}^{\hat{l}j}_{mn} = 0 \tag{3.32c}$$

$$\hat{\theta}^{\hat{j}j}_{mn} = 0 \tag{3.32d}$$

where $\hat{\theta}^{\hat{j}j}_{mn}$ is the extended P-gravitational stress tensor. The extended tracelessness conditions in (3.32c) follow from (3.32a), and are independent of the equations of motion in eq. (3.32d). Not surprisingly, the P-gravitational equations of motion reduce in the conformal gauge (3.31) to the following $2K$ extended Polyakov constraints:

$$\hat{\theta}_{\hat{j}j} = \frac{1}{4\pi}\frac{G^{ab}}{f_j(\sigma)}\sum_{\hat{l}=0}^{f_j(\sigma)-1} \hat{J}_{\hat{l}aj}\hat{J}_{\hat{j}-\hat{l},bj} = 0 \tag{3.33a}$$

$$\hat{\bar{\theta}}_{\hat{j}j} = \frac{1}{4\pi}\frac{G^{ab}}{f_j(\sigma)}\sum_{\hat{l}=0}^{f_j(\sigma)-1} \hat{\bar{J}}_{\hat{l}aj}\hat{\bar{J}}_{\hat{j}-\hat{l},bj} = 0 \tag{3.33b}$$

$$\hat{J}_{\hat{j}aj} = \frac{f_j(\sigma)}{2}G_{ab}\partial_+\hat{x}^{-\hat{j},bj}, \;\; \hat{\bar{J}}_{\hat{j}aj} = -\frac{f_j(\sigma)}{2}G_{ab}\partial_-\hat{x}^{-\hat{j},bj} \tag{3.33c}$$

$$\partial_-\hat{J}_{\hat{j}aj} = \partial_+\hat{\bar{J}}_{\hat{j}aj} = \partial_-\hat{\theta}_{\hat{j}j} = \partial_+\hat{\bar{\theta}}_{\hat{j}j} = 0 \tag{3.33d}$$

$$\bar{\hat{j}} = 0,\ldots,f_j(\sigma)-1, \;\; \sum_j f_j(\sigma) = K, \;\; a,b = 1,\ldots,d. \tag{3.33e}$$

These constraints are independent of the Weyl degrees of freedom, and are indeed nothing but the coordinate-space form of the extended Virasoro constraints (3.8) of the Hamiltonian formulation. I remind that these $2K$ constraints include the two ordinary Polyakov constraints

$$\sum_j \hat{\theta}_{0j} = \sum_j \hat{\bar{\theta}}_{0j} = 0 \tag{3.34}$$

which involve only the total left- and right-mover physical stress tensors of each orbifold sector.

3.4 Extended nambu actions

For each twisted sector σ of the permutation orbifolds $\mathrm{U}(1)^{Kd}/H(\text{perm})$, I also mention the extended action of Nambu-type:

$$\hat{\hat{S}}_\sigma \approx \int dt \int_0^{2\pi} d\xi \sum_j \hat{\hat{H}}^{(0)j} \tag{3.35a}$$

$$\hat{\hat{H}}^{(\hat{\jmath})j} \equiv \frac{1}{f_j(\sigma)} \sum_{J=0}^{f_j(\sigma)-1} e^{-2\pi i \frac{\hat{\jmath}J}{f_j(\sigma)}} \left(-\det\left(\sum_{\hat{l}=0}^{f_j(\sigma)-1} e^{2\pi i \frac{J\hat{l}}{f_j(\sigma)}} \hat{\mathcal{L}}_{mn}^{\hat{l}j} \right)\right)^{\frac{1}{2}} \tag{3.35b}$$

$$\hat{\mathcal{L}}_{mn}^{\hat{\jmath}j} = \frac{1}{8\pi} \sum_{\hat{l}=0}^{f_j(\sigma)-1} \partial_m \hat{x}^{\hat{l}aj} f_j(\sigma) G_{ab} \partial_n \hat{x}^{\hat{\jmath}-\hat{l},bj}. \tag{3.35c}$$

These systems are constructed using the quantity $\hat{\mathcal{L}}_{mn}^{\hat{\jmath}j}$ in eq. (3.35c) as the induced, extended world-sheet metric, which transforms like the extended Polyakov metric:

$$\hat{\mathcal{L}}_{mn}^{\hat{\jmath}j}(\xi+2\pi,t) = e^{2\pi i \frac{\hat{\jmath}}{f_j(\sigma)}} \hat{\mathcal{L}}_{mn}^{\hat{\jmath}j}(\xi,t) \tag{3.36a}$$

$$\delta\hat{\mathcal{L}}_{mn}^{\hat{\jmath}j} = \sum_{\hat{l}=0}^{f_j(\sigma)-1} \left(\hat{\beta}^{p\hat{l}j} \partial_p \hat{\mathcal{L}}_{mn}^{\hat{\jmath}-\hat{l},j} + \partial_m \hat{\beta}^{p\hat{l}j} \hat{\mathcal{L}}_{pn}^{\hat{\jmath}-\hat{l},j} + \partial_n \hat{\beta}^{p\hat{l}j} \hat{\mathcal{L}}_{pm}^{\hat{\jmath}-\hat{l},j} \right). \tag{3.36b}$$

The transformations (3.36) follow from the monodromies (3.20c) and extended diffeomorphisms (3.22a) of the extended coordinates $\{\hat{x}^{\hat{\jmath}aj}, a = 1,\ldots,d\}$, and imply in turn that the quantity $\hat{\hat{H}}^{(\hat{\jmath})j}$ transforms (like $\hat{H}^{(\hat{\jmath})j}$ in (3.27)) as a set of extended scalar densities.

In particular, one finds that

$$\hat{\hat{H}}^{(0)j}(\xi+2\pi,t) = \hat{\hat{H}}^{(0)j}(\xi,t), \qquad \sum_j f_j(\sigma) = K \tag{3.37a}$$

$$\delta\hat{\hat{H}}^{(0)j} = \partial_p \left(\sum_{\hat{\jmath}=0}^{f_j(\sigma)-1} \hat{\beta}^{p\hat{\jmath}j} \hat{\hat{H}}^{(0)j} \right) \tag{3.37b}$$

$$\delta\hat{\hat{S}}_\sigma = 0 \tag{3.37c}$$

so the extended actions of Nambu-type are properly invariant under monodromy and the general permutation-twisted diffeomorphisms.

In Minkowski target space $G = \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1\!\!1 \end{smallmatrix}\right)$, the $N_* = 2K$ degrees of freedom of the extended diffeomorphisms should allow us to follow ref. [66] in choosing the following K gauge conditions as an *extended light-cone gauge*

$$\hat{x}^{\hat{\jmath},+,j} = 0, \qquad \bar{\hat{\jmath}} = 0,\ldots,f_j(\sigma)-1, \quad \sum_j f_j(\sigma) = K \tag{3.38}$$

and show that the K longitudinal coordinates $\hat{x}^{\hat{\jmath},-,j} = \hat{x}^{j,-,j}(\hat{x}^{\perp})$ are dependent on the remaining $K(d-2)$ transverse twisted coordinates $\{\hat{x}^{\hat{\jmath}\alpha j}, \alpha = 1,\ldots,d-2\}$. Correspondingly, the extended light-cone gauge for the Nambu-like action (2.58) of the orientation-orbifold sectors should give a description of the twisted open strings in terms of $2(d-2)$ transverse degrees of freedom.

Subsection 4.1 includes some remarks on the corresponding operator theories in critical dimension $d = 26$.

3.5 A complementary derivation

Following our discussion in subsection 2.9 we are led to consider, in each sector of each permutation orbifold, a corresponding change of variable to the J-basis fields $\hat{A}^J, \hat{A}_J$ with twisted boundary conditions [29, 37]:

$$\hat{x}^{\hat{\jmath}aj} = \frac{1}{f_j(\sigma)} \sum_{J=0}^{f_j(\sigma)-1} e^{-2\pi i \frac{\hat{\jmath}J}{f_j(\sigma)}} \hat{x}^{Jaj} \tag{3.39a}$$

$$\hat{h}_{mn}^{(\hat{\jmath})j} = \frac{1}{f_j(\sigma)} \sum_{J=0}^{f_j(\sigma)-1} e^{-2\pi i \frac{\hat{\jmath}J}{f_j(\sigma)}} \hat{h}_{mn}^{Jj} \tag{3.39b}$$

$$\hat{h}^{mn}_{(\hat{\jmath})j} = \frac{1}{f_j(\sigma)} \sum_{J=0}^{f_j(\sigma)-1} e^{2\pi i \frac{\hat{\jmath}J}{f_j(\sigma)}} \hat{h}^{mn}_{Jj} \tag{3.39c}$$

$$\hat{H}^{(\hat{\jmath})j} = \frac{1}{f_j(\sigma)} \sum_{J=0}^{f_j(\sigma)-1} e^{-2\pi i \frac{\hat{\jmath}J}{f_j(\sigma)}} \sqrt{-\det(\hat{h}_{mn}^{Jj})} \tag{3.39d}$$

$$\hat{h}_{mp}^{Jj} \hat{h}^{pn}_{Jj} = \delta_m^n, \quad J = 0, \ldots, f_j(\sigma) - 1, \quad \sum_j f_j(\sigma) = K. \tag{3.39e}$$

Here J is Fourier-conjugate to $\hat{\jmath}$, and the new fields are periodic $J \to J \pm f_j(\sigma)$, but the monodromies of the new fields are *not diagonal*

$$\hat{A}^{Jj} = \sum_{\hat{\jmath}=0}^{f_j(\sigma)-1} e^{2\pi i \frac{J\hat{\jmath}}{f_j(\sigma)}} \hat{A}^{(\hat{\jmath})j}, \quad \hat{A}_{Jj} = \sum_{\hat{\jmath}=0}^{f_j(\sigma)-1} e^{-2\pi i \frac{J\hat{\jmath}}{f_j(\sigma)}} \hat{A}_{(\hat{\jmath})j} \tag{3.40a}$$

$$\hat{A}^{Jj}(\xi + 2\pi, t) = \hat{A}^{J+1,j}(\xi, t), \quad \hat{A}_{Jj}(\xi + 2\pi, t) = \hat{A}_{J+1,j}(\xi, t). \tag{3.40b}$$

After some algebra, one finds that the general extended action (3.28) of Polyakov-type takes the following simple form in the J-basis

$$\hat{S}_\sigma = \int dt \int_0^{2\pi} d\xi \, \frac{1}{8\pi} \sum_j \sum_{\hat{\jmath},\hat{k},\hat{l}=0}^{f_j(\sigma)-1} \hat{H}^{(\hat{\jmath})j} \hat{h}^{mn}_{(\hat{\jmath}+\hat{k}+\hat{l})j} \partial_m \hat{x}^{\hat{k}aj} f_j(\sigma) G_{ab} \partial_n \hat{x}^{\hat{l}bj} \tag{3.41a}$$

$$= \int dt \int_0^{2\pi} d\xi \, \frac{1}{8\pi} \sum_j \sum_{J=0}^{f_j(\sigma)-1} \sqrt{-\det(\hat{h}^{Jj})} \, \hat{h}^{mn}_{Jj} \partial_m \hat{x}^{Jaj} G_{ab} \partial_n \hat{x}^{Jbj} \tag{3.41b}$$

for each sector σ of each permutation orbifold. Because of the sum on J, the action density in eq. (3.41b) is still manifestly monodromy-invariant under the non-diagonal monodromies of the J-basis fields. Moreover — except for global coupling via the non-diagonal monodromies — the form (3.41b) is a sum of K ordinary untwisted Polyakov actions [64] for the original closed-string CFT $\mathrm{U}(1)^{Kd}$, written now in the cycle basis of each representative element $h_\sigma \in H(\text{perm})$.

This result allows us to understand the general extended action (3.41a) of Polyakov-type as nothing but the form obtained by monodromy- decomposition of the J-basis fields $\hat{A}^{Jj}, \hat{A}_{Jj}$, an essentially standard application of the principle of local isomorphisms [27, 29, 31, 32, 35, 37]. Using only eq. (3.39a) for the twisted coordinates $\{\hat{x}^{\hat{\jmath}aj}\}$, the extended actions (3.35) of Nambu-type can similarly be understood as the monodromy-resolved form of a sum of K ordinary untwisted Nambu actions [65] for the mixed-monodromy coordinates $\{\hat{x}^{Jaj}\}$.

Another useful form of the J-basis action (3.41b) is

$$\hat{S}_\sigma = \sum_j \sum_{J=0}^{f_j(\sigma)-1} \int dt_{Jj} \int_0^{2\pi} d\xi_{Jj}\, \frac{1}{8\pi} \sqrt{-\det(\hat{h}^{Jj}_{mn})}\; \hat{h}^{mn}_{Jj} \partial_m \hat{x}^{Jaj} G_{ab} \partial_n \hat{x}^{Jbj} \tag{3.42}$$

where I have relabelled the dummy variables $\{t, \xi \to t_{Jj}, \xi_{Jj}\}$ separately in each of the K terms. Then we see that the action of sector σ is locally invariant under the product of K diffeomorphism groups

$$\xi^{m\prime}_{Jj} = \xi^{m\prime}_{Jj}(\{\xi^p_{Jj}\}), \quad m = 0, 1,\; J = 0, \dots, f_j(\sigma) - 1,\; \sum_j f_j(\sigma) = K \tag{3.43a}$$

$$\hat{x}^{Jaj\prime}(\{\xi^{p\prime}_{Jj}\}) = \hat{x}^{Jaj}(\{\xi^p_{Jj}\}), \quad a = 0, \dots, d-1 \tag{3.43b}$$

$$\hat{h}^{Jj\prime}_{mn}(\{\xi^{p\prime}_{Jj}\}) = \frac{\partial \xi^p_{Jj}}{\partial \xi^{m\prime}_{Jj}} \frac{\partial \xi^q_{Jj}}{\partial \xi^{n\prime}_{Jj}} \hat{h}^{Jj}_{pq}(\{\xi^p_{Jj}\}) \tag{3.43c}$$

which are *globally intertwined* by the non-diagonal monodromies of the J-basis.

In principle, all the invariances of the J-basis can be mapped back into the twisted $\hat{\jmath}$-basis via the monodromy-decompositions (3.39). I illustrate this here with the simple case of the extended Weyl invariance, again leaving the finite form of the twisted diffeomorphism groups for future work.

The Weyl invariance of $\hat{S}_\sigma$ in the J-basis is

$$\hat{h}^{Jj}_{mn} \to e^{-\hat{\sigma}_{Jj}} \hat{h}^{Jj}_{mn}, \quad \hat{\sigma}_{Jj}(\xi + 2\pi, t) = \hat{\sigma}_{J+1,j}(\xi, t). \tag{3.44}$$

Then, using the monodromy decomposition (3.39) of the extended metric and

$$\hat{\sigma}_{(\hat{\jmath})j} \equiv \frac{1}{f_j(\sigma)} \sum_{J=0}^{f_j(\sigma)-1} e^{2\pi i \frac{\hat{\jmath} J}{f_j(\sigma)}} \hat{\sigma}_{Jj} \tag{3.45}$$

one finds the extended, twisted Weyl invariance in the monodromy-resolved $\hat{\jmath}$-basis:

$$\hat{h}^{(\hat{\jmath})j}_{mn} \to \frac{1}{f_j(\sigma)} \sum_{J,\hat{k}=0}^{f_j(\sigma)-1} \exp\left(2\pi i \frac{J(\hat{k} - \hat{\jmath})}{f_j(\sigma)} - \sum_{\hat{l}=0}^{f_j(\sigma)-1} e^{-2\pi i \frac{J\hat{l}}{f_j(\sigma)}} \hat{\sigma}_{(\hat{l})j}\right) \hat{h}^{(\hat{k})j}_{mn} \tag{3.46a}$$

$$\hat{\sigma}_{(\hat{\jmath})j}(\xi + 2\pi, t) = e^{-2\pi i \frac{\hat{\jmath}}{f_j(\sigma)}} \hat{\sigma}_{(\hat{\jmath})j}(\xi, t) \tag{3.46b}$$

$$\tilde{h}^{mn}_{(\hat{\jmath})j} \to \tilde{h}^{mn}_{(\hat{\jmath})j}, \quad \hat{S}_\sigma \to \hat{S}_\sigma. \tag{3.46c}$$

Similarly, the standard conformal-gauge parametrization of the metric in the J-basis

$$\hat{h}^{Jj}_{mn} = \eta_{mn} e^{-\hat{\phi}_{Jj}}, \quad \hat{\phi}_{Jj}(\xi + 2\pi, t) = \hat{\phi}_{J+1,j}(\xi, t) \tag{3.47a}$$

$$\hat{\phi}_{(\hat{\jmath})j} \equiv \frac{1}{f_j(\sigma)} \sum_{J=0}^{f_j(\sigma)-1} e^{2\pi i \frac{\hat{\jmath} J}{f_j(\sigma)}} \hat{\phi}_{Jj} \tag{3.47b}$$

gives the promised conformal-gauge result (3.31) in terms of the K extended Weyl fields $\{\hat{\phi}_{(\bar{j})j}\}$ with diagonal monodromy.

As a simple example consider the case of $\mathbb{Z}_2$-permutation gravity, for which the monodromy-resolved results above read:

$$\hat{h}_{mn}^{(0)} \to e^{-\hat{\sigma}_{(0)}}(\hat{h}_{mn}^{(0)}\cos(\hat{\sigma}_{(1)}) - i\hat{h}_{mn}^{(1)}\sin(\hat{\sigma}_{(1)})) \tag{3.48a}$$

$$\hat{h}_{mn}^{(1)} \to e^{-\hat{\sigma}_{(0)}}(\hat{h}_{mn}^{(1)}\cos(\hat{\sigma}_{(1)}) - i\hat{h}_{mn}^{(0)}\sin(\hat{\sigma}_{(1)})) \tag{3.48b}$$

$$\hat{h}_{mn}^{(0)} = \eta_{mn}e^{-\hat{\phi}_{(0)}}\cos(\hat{\phi}_{(1)}), \quad \hat{h}_{mn}^{(1)} = -i\eta_{mn}e^{-\hat{\phi}_{(0)}}\sin(\hat{\phi}_{(1)}) \tag{3.48c}$$

$$\hat{\sigma}_{(u)}(\xi+2\pi,t) = (-1)^u\hat{\sigma}_{(u)}(\xi,t), \quad \hat{\phi}_{(u)}(\xi+2\pi,t) = (-1)^u\hat{\phi}_{(u)}(\xi,t) \tag{3.48d}$$

$$\hat{h}_{mn}^{(u)}(\xi+2\pi,t) = (-1)^u\hat{h}_{mn}^{(u)}(\xi,t). \tag{3.48e}$$

Here I have suppressed the single cycle index $j = 0$, and relabelled $\bar{\hat{j}} = \bar{u} = 0, 1$. These are the same extended Weyl transformations (2.62b) and conformal-gauge extended metric (2.51b) found for $\mathbb{Z}_2$-permutation gravity in the open-string orientation-orbifold sectors, except that those results were left in terms of the fields with non-diagonal monodromy $(0 \leftrightarrow 1)$

$$\hat{\sigma}_0 = \hat{\sigma}_{(0)} + \hat{\sigma}_{(1)}, \; \hat{\sigma}_1 = \hat{\sigma}_{(0)} - \hat{\sigma}_{(1)}, \; \hat{\phi}_0 = \hat{\phi}_{(0)} + \hat{\phi}_{(1)}, \; \hat{\phi}_1 = \hat{\phi}_{(0)} - \hat{\phi}_{(1)} \tag{3.49}$$

which would be inappropriate for the final form of the permutation orbifolds.

4. Discussion

4.1 The conjecture: physical strings at higher central charge

Based on their extended (twisted) Virasoro algebras [27, 55, 35], I have found extended world-sheet action formulations, of both Polyakov- and Nambu-type, in the twisted sectors of the orbifolds of permutation-type. This includes in particular the twisted open-string sectors at $\hat{c} = 2d$ of the orientation orbifolds $\mathrm{U}(1)^d/(\mathbb{Z}_2(w.s.) \times H)$ and the twisted closed-string sectors at $\hat{c} = Kd$ of the permutation orbifolds $\mathrm{U}(1)^{Kd}/H(\text{perm})$, where in both cases $\mathrm{U}(1)^d$ is the d-dimensional free-bosonic closed-string CFT. The extended actions of Polyakov-type exhibit a class of new extended (twisted) world-sheet gravities called the *permutation gravities*, which are classified by the conjugacy classes of the permutation groups. Both the Polyakov- and the Nambu-type actions are invariant under the extended, twisted diffeomorphism groups generated by the extended Virasoro algebras.

In the covariant formulation of the corresponding Minkowski-space quantum theories, each of these twisted sectors carries an increased number of *negative-norm states* (ghosts) corresponding to the higher central charge. The negative norms are associated as usual with the number of twisted time-like currents of each sector:

$$N_* = \begin{cases} 2 \text{ for open-string orient.orb.sectors} \\ 2K \text{ for closed-string perm.orb.sectors.} \end{cases} \tag{4.1}$$

For example, the $2K$ twisted time-like currents of the permutation orbifolds have the form

$$\partial_\pm \hat{x}^{\hat{j}0j}, \quad \bar{\hat{j}} = 0,1,\ldots,f_j(\sigma)-1, \quad \sum_j f_j(\sigma) = K \tag{4.2a}$$

$$G_{ab} = \begin{pmatrix} -1 & 0 \\ 0 & 1\!\!1 \end{pmatrix}, \quad a = 0,1,\ldots,d-1 \tag{4.2b}$$

and (although it is masked by the unitary transformation (2.23) of the Minkowski metric) the ghost-doubling in (4.1) for the orientation orbifolds follows in the same way from the doubling $\bar{u} = 0,1$ of the extended coordinates. In both cases, the counting (4.1) of time-like currents holds as well for the untwisted sector of the orbifolds — where the doubling for the orientation orbifolds counts both the left- and right-mover currents of the untwisted closed-string CFT.

But we have seen the number N_* earlier in our discussion. It is also the number of degrees of freedom in the extended diffeomorphisms of each twisted sector, or equivalently the number of extended Virasoro (Polyakov) constraints (see eqs. (2.56) and (3.9)) in each sector. This allows us to conjecture that the orbifolds of permutation-type define operator-string theories which are free of negative-norm states at higher central charge. In particular, we may expect that the extended Virasoro constraints of the classical theories will translate into *extended physical-state conditions* associated to extended Ward identities for the twisted tree amplitudes and loops of the corresponding orbifold-string theories.

More precisely, consideration of orbifold-string loops and/or BRST operators for the twisted sectors should fix the critical dimension of the ghost-free theories at $d = 26$, so that the critical orbifold-string theories

$$\frac{\mathrm{U}(1)^{26}}{\mathbb{Z}_2(w.s.)\times H}, \quad \hat{c} = 52 \text{ for open-string sectors} \tag{4.3a}$$

$$\frac{\mathrm{U}(1)^{26K}}{H(\text{perm})}, \quad \hat{c} = 26K \text{ for all sectors} \tag{4.3b}$$

are orbifolds of decoupled copies of the critical free-bosonic string. This picture makes sense because orbifolding should not create negative-norm states when the original symmetric string theory was completely physical.

In the succeeding papers of this series, I will augment this intuition by constructing the twisted BRST systems of $\hat{c} = 52$ matter, as well as new extended Ward identities, ghost-free tree amplitudes and modular-invariant loops for the permutation orbifold-strings. I will also be able to shed some light on the somewhat mysterious relation between orientation orbifolds and conventional orientifolds.

The extended actions of Nambu-type offer another approach to this conjecture. In this case one can choose an extended light-cone gauge (see Subsection 3.4) in each twisted sector, which should then show effective central charges

$$\hat{c}_{\text{eff}} = \begin{cases} 52-4 = (26-2)\cdot 2 = 48 & \text{(orientation orbs)} \\ (26-2)K = 24K & \text{(permutation orbs)} \end{cases} \tag{4.4}$$

for the physical (transverse) degrees of freedom in the critical orbifolds. Following ref. [66] we know that consistency of this quantization will involve space-time interpretation of the extended coordinates $\hat{x}$ (52 or 26+26 ?), to be determined by closure of the appropriate space-time group generators in each twisted sector.

4.2 Other orbifolds of permutation-type

Our classical discussion of the orbifolds of permutation-type is by no means complete. Beyond the issues left unfinished here for the orientation and permutation orbifolds, many other orbifolds of permutation-type are known, whose extended formulation can be studied with the techniques developed here:

- The *generalized* free-bosonic permutation orbifolds [45, 46]

$$\frac{\mathrm{U}(1)^{26K}}{H_+}, \quad H_+ = H(\mathrm{perm}) \times H' \tag{4.5}$$

at critical central charge $\hat{c} = 26K$. Here the group H_+ can involve extra automorphisms H' which act uniformly on each closed-string copy $\mathrm{U}(1)^{26}$. Following the reasoning above, I present only the initial data and the final form of the extended action for each twisted sector of the generalized permutation orbifolds:

$$J_{aI}{}' = P_I{}^J \omega_a{}^b J_{bJ}, \;\; \bar{J}_{aI}{}' = P_I{}^J \omega_a{}^b \bar{J}_{bJ} \tag{4.6a}$$

$$P \in H(\mathrm{perm}), \quad \omega \in H' \tag{4.6b}$$

$$\omega_a{}^b (U^\dagger)_b{}^{n(r)\mu} = (U^\dagger)_a{}^{n(r)\mu} e^{-2\pi i \frac{n(r)}{\rho(\sigma)}} \tag{4.6c}$$

$$\mathcal{G}_{n(r)\mu;n(s)\nu} \equiv \chi_{n(r)\mu} \chi_{n(s)\nu} U_{n(r)\mu}{}^a U_{n(s)\nu}{}^b G_{ab} \tag{4.6d}$$

$$= \delta_{n(r)+n(s),0 \bmod \rho(\sigma)} \mathcal{G}_{n(r)\mu;-n(r)\nu} \tag{4.6e}$$

$$\hat{S}_\sigma = \frac{1}{8\pi} \int dt \int_0^{2\pi} d\xi \;\times \tag{4.6f}$$

$$\times \sum_j f_j(\sigma) \sum_{\hat{j},\hat{k},\hat{l}=0}^{f_j(\sigma)-1} \hat{H}^{(\hat{j})j} \hat{h}^{mn}_{(\hat{j}+\hat{k}+\hat{l})j} \partial_m \hat{x}^{\hat{k}n(r)\mu j} \mathcal{G}_{n(r)\mu;n(s)\nu} \partial_n \hat{x}^{\hat{l}n(s)\nu j}$$

$$\hat{x}^{\hat{j}n(r)\mu j}(\xi+2\pi, t) = e^{-2\pi i \left(\frac{\hat{j}}{f_j(\sigma)} + \frac{n(r)}{\rho(\sigma)} \right)} \hat{x}^{\hat{j}n(r)\mu j}(\xi, t) \tag{4.6g}$$

$$\bar{\hat{j}} = 0, 1, \ldots, f_j(\sigma) - 1, \quad \sum_j f_j(\sigma) = K, \quad \bar{n}(r) \in (0, 1, \ldots, \rho(\sigma) - 1). \tag{4.6h}$$

In these cases, the twisted metric $\mathcal{G}$ in eq. (4.6d) is constructed from the solution of the H-eigenvalue problem (4.6c) of each extra automorphism $\omega \in H'$. Note in particular that, as seen in the extended actions (4.6f), the generalized permutation orbifolds involve the *same* twisted permutation gravities studied above.

We have seen such a universality before in the case of the $\mathbb{Z}_2$-permutation gravity of all the orientation orbifolds. Indeed the permutation gravities couple universally to the extended stress tensors, which are associated to the same orbifold Virasoro algebras (3.2) in each twisted sector of any orbifold of permutation-type. The orbifold Virasoro algebras are themselves universal because they arise [35, 43] simply by twisting the action of $H(\text{perm})$ on the ordinary decoupled Virasoro copies in the untwisted sector of the orbifold — independent of the action of any particular H' on a given copy. For the generalized permutation orbifolds in eq. (4.6), the explicit form of their "doubly-twisted" current algebras (with modeing $J_{\hat{j}n(r)\mu j}(m+\frac{\hat{j}}{f_j(\sigma)}+\frac{n(r)}{\rho(\sigma)})$ and (conformal gauge) extended stress tensors can be obtained by the substitution $\{a \to n(r)\mu, G \to \mathcal{G}\}$ in eqs. (3.14a,d,e).

- The free-bosonic *open-string permutation orbifolds* [45]

$$\frac{\mathrm{U}(1)^{26K}_{\text{open}}}{H(\text{perm})} \tag{4.7}$$

and their T-dualizations [46] at critical central charge $\hat{c} = 26K$, which will also be governed by the same permutation gravities. Starting from the left-mover data of the generalized (closed-string) permutation orbifolds above, the generalized open-string permutation orbifolds

$$\frac{\mathrm{U}(1)^{26K}_{\text{open}}}{H(\text{perm}) \times H'} \tag{4.8}$$

and their T-duals at $\hat{c} = 26K$ can also be worked out as special cases of ref. [46]. (The sectors of the generalized open-string $\mathbb{Z}_2$-permutation orbifolds are T-dual to the open-string sectors of the orientation orbifolds.)

- The *superstring orbifolds of permutation-type* at critical central charge $\hat{c} = 10K$ (and $(\hat{c}, \hat{\bar{c}}) = (26K, 10K)$ for heterotic type). The first goal here will be the explicit form of the world-sheet *permutation supergravities*, associated to the extended, twisted superconformal algebras [22, 27] of these orbifolds.

- The *partial orbifoldizations* of permutation-type, for example

$$\frac{\mathrm{U}(1)^{26} \times \mathrm{U}(1)^{26}}{\mathbb{Z}_2(D)}, \quad 0 \le D \le 25, \quad \hat{c} = 52 \tag{4.9}$$

where $\mathbb{Z}_2(D)$ exchanges only two subsets $\{D\}$ of D bosons each. The single twisted sector of this orbifold is described by a *hybrid* action

$$\hat{S} = S_1(26 - D) + S_2(26 - D) + \hat{S}(2D) \tag{4.10}$$

where $S_{1,2}$ are ordinary Polyakov actions for $26 - D$ untwisted bosons and $\hat{S}(2D)$ is the extended action (3.28) with $\mathbb{Z}_2$-twisted permutation gravity coupled to $2D$ twisted bosons

$$\{\hat{x}^{\hat{j}a0}, , \forall\ a \in \{D\},\ \bar{\hat{j}} = 0, 1\}. \tag{4.11}$$

Extensions to higher genus, as well as non-trivial B fields and twisted B fields [37] can also be studied.

Our discussion above suggests that all the critical orbifold CFT's of permutation-type can describe twisted physical string systems at higher central charge.

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